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Azar y Autómatas

Clase 3

Casi todas las secuencias son normales

Equivalencia entre las definiciones de normalidad

Lema de Piatetski-Shapiro

Notation

Number of occurrences, not-aligned and aligned,

$$
\begin{aligned}
|w|_u &= |\{i : w[i \ldots i + |u| - 1] = u\}|, \\
\|w\|_u &= |\{i : w[i \ldots i + |u| - 1] = u \text{ and } i \equiv 1 \mod |u|\}|.
\end{aligned}
$$

For example, $|aaaaa|_{aa} = 4$ and $\|aaaaa\|_{aa} = 2$.

Notice that the definition of aligned occurrences has the condition $i \equiv 1 \mod |u|$ instead of $i \equiv 0 \mod |u|$, because the positions are numbered starting at 1.

When a word u is just a symbol, $|w|_u$ and $\|w\|_u$ coincide.

Counting aligned occurrences

Aligned occurrences of a word of length r over alphabet A coincide with occurrences of the corresponding symbol over alphabet A^r.

Consider alphabet A, a length r and alphabet B with $|A|^r$ symbols.
The set of words of length r over alphabet A and the set B are isomorphic:

$$\pi : A^r \to B$$

induced by the lexicographic order in the respective sets.

Thus, for any $w \in A^*$ such that $|w|$ is a multiple of r,

$$|\pi(w)| = |w|/r.$$

Then,

$$\forall\ u \in A^r\ (\|w\|_u = |\pi(w)|_{\pi(u)}).$$

Representation of real numbers

A *base* is an integer greater than or equal to 2. For a positive real number x, the *expansion* of x in base b is a sequence $a_1 a_2 a_3 \ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k \geq 1} a_k b^{-k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

To have a unique representation of all rational numbers we require that expansions do not end with a tail of $b-1$.

We will abuse notation and whenever the base b is understood we will denote the first n digits in the expansion of x with $x[1 \ldots n]$.

Definition of normality

Definition 1 (Strong aligned normality, Borel 1909)

A real number x is **simply normal to base b** if, in the expansion of x in base b, each digit d occurs with limiting frequency equal to $1/b$,

$$\lim_{n\to\infty} \frac{|x[1\ldots n]|_d}{n} = \frac{1}{b}$$

A real number x is **normal to base b** if each of the reals $x, bx, b^2x, \ldots$ are simply normal to bases $b^1, b^2, b^3, \ldots$.

A real x is **absolutely normal** if x is normal to every integer base greater than or equal to 2.

Equivalences between combinatorial definitions of normality

A real number x is simply normal to base b if, in the expansion of x in base b, each digit d occurs with limiting frequency equal to $1/b$.

Borel's original definition of normality turned out to be redundant.

Definition (Strong aligned normality, Borel 1909)

A real number x is normal to base b if each of the reals $x, bx, b^2x, \ldots$ are simply normal to bases $b^1, b^2, b^3, \ldots$

Definition (Aligned normality, Pillai 1940)

A real number x is normal to base b if x is simply normal to bases $b^1, b^2, b^3, \ldots$.

Definition (Non-aligned normality, Borel first: Niven and Zuckerman 1951)

A real number x is normal to base b if for every block u,

$$\lim_{n\to\infty} \frac{|x[1 \ldots n]|_u}{n} = \frac{1}{b^{|u|}}.$$

A central limit theorem: there are a few bad words

Let A be an alphabet of b symbols, let k be a positive integer and let ε be a real number between 0 and 1.

We define the set of words of length k such that a given word w has a number of occurrences that differs from the expected in plus or minus εk,

$$Bad(A, k, w, \varepsilon) = \left\{ v \in A^k : \left| |v|_w - \frac{k}{b^{|w|}} \right| > \varepsilon k \right\}.$$

Example: For $A = \{0, 1\}$, $k = 4$, $\varepsilon = 1/4$, $w = 11$,
we have $\frac{k}{b^{|w|}} = \frac{4}{2^2} = 1$, $\epsilon k = 1$.
$Bad(A, k, w, \varepsilon) = \{1111\}$ the set of words with 3 occurences of w:

For $A = \{0, 1\}$, $k = 4$, $\varepsilon = 1/4$,$w = 1$, we have $\frac{k}{b^{|w|}} = \frac{4}{2^1} = 2$, $\epsilon k = 1$,
$Bad(A, k, w, \varepsilon) =$ the set of words with $4, 0$ occurences of w:

$$\{1111, 0000\}$$

There are a few bad words

Lemma 1 (Adapted from Hardy and Wright's book, Theorem 148)

Let b be an integer such that $b \geq 2$ and let k be a positive integer. If $6/k \leq \varepsilon \leq 1/b$ then for every symbol d in A,

$$|Bad(A, k, d, \varepsilon)| < 4e^{-b\varepsilon^2 k/6} b^k.$$

This lemma is known in many different variants, such as Bernstein's inequality where the is no constraint on ε.

Proof of Lemma 1

Observe that for any symbol d in A, the number of words of lengh k having exactly n occurrences of a given digit d is :

$$\binom{k}{n}(b-1)^{k-n}$$

Then,

$$|Bad(A,k,d,\varepsilon)| = \sum_{n \le k/b - \varepsilon k} \binom{k}{n}(b-1)^{k-n} + \sum_{n \ge k/b + \varepsilon k} \binom{k}{n}(b-1)^{k-n}$$

Fix b and k and write $N(n)$ for $\binom{k}{n}(b-1)^{k-n}$.

For all $n < k/b$, $N(n) < N(n+1)$ and these quotients increase as n increases,

$$\frac{N(n)}{N(n+1)} = \frac{(n+1)(b-1)}{k-n}$$

Similarly, for all $n > k/b$, $N(n) < N(n-1)$ and these quotients increase as n decreases,

$$\frac{N(n)}{N(n-1)} = \frac{k-n+1}{n(b-1)}$$

We will "shift" m positions each of the sums for $|Bad(A,k,d,\varepsilon)|$.
We bound the first sum as follows. Let

$$m = \lfloor \varepsilon k/2 \rfloor \text{ and } p = \lfloor k/b - \varepsilon k \rfloor$$

For any n we can write

$$N(n) = \frac{N(n)}{N(n+1)} \cdot \frac{N(n+1)}{N(n+2)} \cdot \ldots \cdot \frac{N(n+m-1)}{N(n+m)} \cdot N(n+m)$$

For each n such that $n \leq p+m-1$ we have that $n+m < k/b$, so,

$$\begin{aligned}
\frac{N(n)}{N(n+1)} &\leq \frac{N(p+m-1)}{N(p+m)} = \frac{(p+m)(b-1)}{k-p-m+1} \\
&< \frac{(k/b - \varepsilon k/2)(b-1)}{k - k/b + \varepsilon k/2} = 1 - \frac{\varepsilon b/2}{1 - 1/b + \varepsilon/2} \\
&< 1 - \varepsilon b/2 \qquad \text{(using the hypothesis } \varepsilon \leq 1/b\text{)}. \\
&< e^{-b\varepsilon/2}.
\end{aligned}$$

Then,

$$\begin{aligned}
N(n) &< \left(e^{-b\varepsilon/2}\right)^m N(n+m) \\
&\leq e^{-b\varepsilon(\varepsilon k/2-1)/2} N(n+m) \\
&\leq 2e^{-b\varepsilon^2 k/4} N(n+m), \qquad \text{(the hypothesis } \varepsilon \leq 1/b \text{ implies } e^{b\varepsilon/2} < 2\text{)}
\end{aligned}$$

We obtain,

$$\sum_{n \le p} N(n) < 2e^{-b\varepsilon^2 k/2} \sum_{n \le p} N(n+m) \le e^{-b\varepsilon^2 k/4} 2\ b^k.$$

We now bound the second sum. Let

$$m = \lfloor \varepsilon k/2 \rfloor \text{ and } q = \lceil k/b + \varepsilon k \rceil.$$

For any n we can write

$$N(n) = \frac{N(n)}{N(n-1)} \cdot \frac{N(n-1)}{N(n-2)} \cdot \ldots \cdot \frac{N(n-m+1)}{N(n-m)} \cdot N(n-m).$$

For each n such that $n \geq q - m + 1$ we have $n - m > k/b$, so,

$$\begin{aligned}
\frac{N(n)}{N(n-1)} &\leq \frac{N(q-m+1)}{N(q-m)} = \frac{k-q+m}{(q-m+1)(b-1)} \\
&= \frac{k - \lceil k/b + \varepsilon k \rceil + \lfloor \varepsilon k/2 \rfloor}{(\lceil k/b + \varepsilon k \rceil - \lfloor \varepsilon k/2 \rfloor + 1)(b-1)} \\
&\leq \frac{k - k/b - \varepsilon k/2}{(k/b + \varepsilon k/2 + 1)(b-1)} \\
&< \frac{1 - 1/b - \varepsilon/2}{(1/b + \varepsilon/2)(b-1)} \\
&\leq 1 - b\varepsilon/3, \qquad \text{using } \varepsilon \leq 1/b.
\end{aligned}$$

We conclude,

$$\frac{N(n)}{N(n-1)} \quad \leq \quad 1 - b\varepsilon/3 \leq e^{-b\varepsilon/3}.$$

Then,

$$\begin{aligned} N(n) &< \left(e^{-b\varepsilon/3}\right)^m N(n-m) \\ &\leq e^{-b\varepsilon\lfloor \varepsilon k/2 \rfloor/3} N(n-m) \\ &\leq e^{-b\varepsilon(\varepsilon k/2 - 1)/3} N(n-m) \\ &\leq 2\ e^{-b\varepsilon^2 k/6} N(n-m), \qquad \text{(the hypothesis } \varepsilon \leq 1/b \text{ implies } e^{b\varepsilon/3} < 2) \end{aligned}$$

Thus,

$$\sum_{n \geq q} N(n) < 4\ b^k e^{-b\varepsilon^2 k/6}.$$

This completes the proof.□

There are a few bad words

Lemma 2

Let A be an alphabet of b symbols. Let k, ℓ be positive integers and ε a real such that $6/\lfloor k/\ell \rfloor \leq \varepsilon \leq 1/b^\ell$. Then,

$$\left| \bigcup_{w \in A^\ell} Bad(A, k, w, \varepsilon\ell) \right| < 4\ell \ b^{2\ell} \ e^{-b^\ell \varepsilon^2 k/(6\ell)} b^k.$$

Proof of Lemma 2

Split the set $\{0, 1, 2, \ldots, k-1\}$ into the congruence classes modulo ℓ.
Each of these classes contains either $\lfloor k/\ell \rfloor$ or $\lceil k/\ell \rceil$ elements.
Let M_0 denote the class of all indices which leave remainder zero when being reduced modulo ℓ. Let $n_0 = |M_0|$.
For each x in A^k consider the word in $(A^\ell)^{n_0}$

$$x[i_1 \ldots (i_1 + \ell - 1)]x[i_2 \ldots (i_2 + \ell - 1)] \ldots x[i_{n_0} \ldots (i_{n_0} + \ell - 1)]$$

for $i_1, \ldots i_{n_0} \in M_0$.
By Lemma 1,

$$\left| Bad(A^\ell, n_0, w, \varepsilon) \right| < 4 \ (b^\ell)^{n_0} e^{-b^\ell \varepsilon^2 n_0 / 6}.$$

Clearly, similar estimates hold for the indices in the other residue classes.

Let $n_1, \ldots, n_{\ell-1}$ denote the cardinalities of these other residue classes. By assumption $n_0 + \cdots + n_{\ell-1} = k$. Then,

$$
\begin{aligned}
|Bad(A,k,w,\varepsilon\ell)| &\leq \sum_{j=0}^{\ell-1} \left| Bad(A^{\ell}, n_j, w, \varepsilon) \right| \\
&\leq \sum_{j=0}^{\ell-1} 4(b^{\ell})^{n_j} e^{-b^{\ell}\varepsilon^2 n_j/6} \\
&\leq \sum_{j=0}^{\ell-1} 4(b^{\ell})^{k/\ell+1} e^{-b^{\ell}\varepsilon^2 k/(6\ell)} = 4\ \ell\ b^{k+\ell}\ e^{-b^{\ell}\varepsilon^2 k/(6\ell)}.
\end{aligned}
$$

The last inequality holds because

$$(b^{\ell})^{\lceil k/\ell \rceil} e^{-b^{\ell}\varepsilon^2 \lceil k/\ell \rceil/6} < (b^{\ell})^{k/\ell+1} e^{-b^{\ell}\varepsilon^2 k/(6\ell)}$$

and $\varepsilon \leq 1/b^{\ell}$ ensures

$$(b^{\ell})^{\lfloor k/\ell \rfloor} e^{-b^{\ell}\varepsilon^2 \lfloor k/\ell \rfloor/6} \quad \leq b^k e^{-b^{\ell}\varepsilon^2 k/(6\ell)}.$$

Now, summing up over all $w \in A^\ell$ we obtain

$$\left| \bigcup_{w \in A^\ell} Bad(A, k, w, \varepsilon\ell) \right| < 4\ell \; b^{k+2\ell} e^{-b^\ell \varepsilon^2 k/(6\ell)}.$$

□

Theorem 3

Almost all sequences are normal.

Proof of Theorem 3

Fix alphabet A. By definition, a sequence x is normal if for every word w and for every ε there is $\tilde{k}$ such that for every $k \geq \tilde{k}$, $x[1 \ldots k] \notin Bad(A, k, w, \varepsilon)$.

Thus, if x is not normal there is ε_0 and there is a word w such that for every $\tilde{k}$ there is $k \geq \tilde{k}$ such that $x[1 \ldots k] \in Bad(A, k, w, \varepsilon_0)$.

We will show that these Bad sets have very few words. By the following properties of the Bad sets

- If $\delta > \varepsilon$ then $Bad(A, k, w, \delta) \subseteq Bad(A, k, w, \varepsilon)$.
- If z is prefix of w then $Bad(A, k, w, \varepsilon) \subseteq Bad(A, k, z, \varepsilon)$.

we can take decreasing values of ε and shorter witnesses z and show that Bad sets are small enough.

Consider ε a decreasing function of k going to *zero*; such as, $\varepsilon = 1/\sqrt[4]{k}$.
Consider ℓ an increasing function of k, unbounded; such as, $\ell = \lfloor \log k \rfloor$.
Since

$$\left| \bigcup_{w \in A^{\leq \lfloor \log k \rfloor}} Bad(A, k, w, (\log k)/\sqrt[4]{k}) \right| < b^k \sum_{\ell=1}^{\lfloor \log k \rfloor} 4\ell b^{2\ell} \; e^{-b^{\ell} \left(\frac{1}{\sqrt[4]{k}} \right)^2 \frac{k}{6\ell}}$$

$$< b^k e^{-\sqrt{k}}, \qquad \text{for } k \text{ large enough}$$

Then, there is k_0 such that

$$\sum_{k \geq k_0} \left| b^{-k} \bigcup_{w \in A^{\leq \lfloor \log k \rfloor}} Bad(A, k, w, (\log k)/\sqrt[4]{k}) \right|$$

is as small as we want.

The proportion of sequences that have an initial segment in some of the Bad sets shrinkes as much as we want when k_0 increases.
This means that almost all sequences haver their initial segments outside of the Bad sets. This proves the theorem. □

A little trick

Lemma 4

Let $(x_{1,n})_{n\geq 0}, (x_{2,n})_{n\geq 0}, \ldots, (x_{k,n})_{n\geq 0}$ *be sequences of real numbers such that* $\sum_{i=1}^{k} x_{i,n} = 1$ *and let* $c_1, c_2, \ldots, c_k$ *be real numbers such that* $\sum_{i=1}^{k} c_i = 1$*. Then,*

1. *If for each* i, $\liminf_{n\to\infty} x_{i,n} \geq c_i$ *then for each* i, $\lim_{n\to\infty} x_{i,n} = c_i$.
2. *If for each* i, $\limsup_{n\to\infty} x_{i,n} \leq c_i$ *then for each* i, $\lim_{n\to\infty} x_{i,n} = c_i$.

We will aply this for $k = b^\ell, x_{n,i} = \frac{|x[1..n]|_{w_i}}{b^\ell}$ for $i = 1, 2, \ldots b^\ell$.
Notar que $\sum_{i=1}^{b^\ell} \frac{|x[1..n]|_{w_i}}{b^\ell} = 1$

Proof of Lemma 4

For any i in $\{1, \ldots, k\}$,

$$\begin{aligned}
\limsup_{n\to\infty} x_{i,n} &= \limsup_{n\to\infty}(1 - \sum_{j\neq i} x_{j,n}) \\
&= 1 + \limsup_{n\to\infty}(-\sum_{j\neq i} x_{j,n}) \\
&= 1 - \liminf_{n\to\infty}(\sum_{j\neq i} x_{j,n}) \\
&\leq 1 - \sum_{j\neq i} \liminf_{n\to\infty} x_{j,n} \\
&\leq 1 - \sum_{j\neq i} c_j = c_i.
\end{aligned}$$

Since

$$\limsup_{n\to\infty} x_{i,n} \leq c_i \leq \liminf_{n\to\infty} x_{i,n}, \text{ and } \liminf \leq \limsup$$

necessarily,

$$\liminf_{n\to\infty} x_{i,n} = \limsup_{n\to\infty} = c_i \text{ and } \lim_{n\to\infty} x_{i,n} = c_i. \ \square$$

Theorem 5 (Piatetski-Shapiro 1957)

Let x be a real and let b be an integer greater than or equal to 2. Let $A = \{0, \ldots, b-1\}$. The following conditions are equivalent,

1. *The real x is normal to base b.*
2. *There is a constant C such that for infinitely many lengths ℓ and for every w in A^ℓ*

$$\limsup_{n\to\infty} \frac{|x[1\ldots n]|_w}{n} < C \cdot b^{-\ell}.$$

3. *There is a constant C such that for infinitely many lengths ℓ and for every w in A^ℓ*

$$\limsup_{n\to\infty} \frac{\|x[1\ldots n\ell]\|_w}{n} < C \cdot b^{-\ell}.$$

Proof of Theorem 5

We prove $2 \Rightarrow 1$.
Suppose C such that for infinitely many lengths ℓ and for every $w \in A^\ell$,

$$\limsup_{n\to\infty} \frac{|x[1\dots n]|_w}{n} < C \cdot b^{-\ell}.$$

Let ℓ be one of those infinitely many lengths. Fix $\varepsilon \leq 1/b^\ell$. Fix $w \in A^\ell$.
Let k be large enough so that $|Bad(A,k,w,\varepsilon)| < b^k\varepsilon$.
Observe that for every $w \in A^*$, for every n and k,

$$|x[1\dots nk]|_w \geq \frac{1}{k-\ell+1} \sum_{v\in A^k} |x[1\dots nk]|_v |v|_w.$$

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{|x[1\dots nk]|_w}{nk} &\geq \liminf_{n\to\infty} \frac{1}{k-\ell+1} \sum_{v\in A^k} \frac{|x[1\dots nk]|_v}{nk} |v|_w \\
&\geq \liminf_{n\to\infty} \frac{1}{k} \sum_{v\in A^k} \frac{|x[1\dots nk]|_v}{nk} |v|_w \\
&\geq \liminf_{n\to\infty} \sum_{v\in A^k} \frac{|x[1\dots nk|_v}{nk} \; \frac{|v|_w}{k} \\
&\geq \liminf_{n\to\infty} \sum_{v\in A^k\setminus Bad(A,k,w,\varepsilon)} \frac{|x[1\dots nk|_v}{nk} \; \frac{|v|_w}{k} \\
&\geq (1-\varepsilon)b^{-\ell} \; \liminf_{n\to\infty} \sum_{v\in A^k\setminus Bad(A,k,w,\varepsilon)} \frac{|x[1\dots nk|_v}{nk} \\
&= (1-\varepsilon)b^{-\ell} \; \liminf_{n\to\infty} \left(1 - \sum_{v\in Bad(A,k,w,\varepsilon)} \frac{|x[1\dots nk|_v}{nk}\right) \\
&\geq (1-\varepsilon)b^{-\ell} \left(1 - \sum_{v\in Bad(A,k,w,\varepsilon)} \limsup_{n\to\infty} \frac{|x[1\dots nk]|_v}{nk}\right) \\
&\geq (1-\varepsilon)b^{-\ell} \left(1 - \sum_{v\in Bad(A,k,w,\varepsilon)} C\cdot b^{-k}\right) \\
&\geq (1-\varepsilon)b^{-\ell}(1 - C\varepsilon).
\end{aligned}
$$

The previous inequality hods for every positive $\varepsilon \leq 1/b^{\ell}$, hence,

$$\liminf_{n\to\infty} \frac{|x[1\dots nk]|_w}{nk} \geq b^{-\ell}.$$

Finally, this last inequality is true for every $w \in A^{\ell}$, so by Lemma 4

$$\lim_{n\to\infty} \frac{|x[1\dots n]|_w}{n} = b^{-\ell}.$$

□

Theorem 6

The three definitions of normality are equivalent.

Proof of Theorem 6

1. We show that *Strong aligned normality* implies *Non-aligned normality*.

Idea : for any $w \in A^\ell$,

$$|x[1 \ldots n]|_w = \sum_{i=0}^{\ell-1} \|(b^i x)[1 \ldots n-i]\|_w$$

By Strong aligned normality, for $i = 1, 2, \ldots$, for every w, writing $\ell = |w|$,

$$\lim_{n\to\infty} \frac{\|(b^i x)[1 \ldots \ell n]\|_w}{n} = b^{-\ell} \text{ equivalently } \lim_{n\to\infty} \frac{\|(b^i x)[1 \ldots n]\|_w}{n/\ell} = b^{-\ell}$$

$$\text{equivalently } \lim_{n\to\infty} \frac{\|(b^i x)[1 \ldots n]\|_w}{n} = b^{-\ell}/\ell$$

Then,

$$\lim_{n\to\infty} \frac{|x[1 \ldots n]|_w}{n} = \sum_{i=0}^{\ell-1} \lim_{n\to\infty} \frac{\|(b^i x)[1 \ldots n-i]\|_w}{n} = \sum_{i=0}^{\ell-1} \frac{b^{-\ell}}{\ell} = b^{-\ell}.$$

2. We prove that *Non-aligned normality* implies *Aligned normality*.

We first define, for any $w \in A^\ell$, $r = 0, .., \ell - 1$,

$$\|x\|_{w,r} = \Big| \{ i : x[i..i + |w| - 1] i \mod |w| = r \} \Big|$$
$$\|x\|_{w,*} = \max\{ \|x\|_{w,r} : r = 1, .., \ell \}$$

Idea: for any large enough K

$$\|x[1 \dots N]\|_w \leq \frac{1}{K - |w| + 1} \sum_{t=1}^{N-K+1} \|x[t \dots t + K)\|_{w,*}$$
$$\leq \frac{1}{K - |w| + 1} \sum_{v \in A^K} |x[1..N]|_v \|v\|_{w,*}$$

$$\widetilde{Bad}(A, k, w, \varepsilon) = \left\{ v \in A^k : |\|v\|_{w,*} - b^{-\ell} k/\ell| > \varepsilon k/\ell \right\}$$

With an argument similar to the proof of Lemma 2 we obtain that for each ε there is k_0 such that for every $k \geq k_0$,

$$|\widetilde{Bad}(A, k, w, \varepsilon)| < \varepsilon b^k$$

Assume the previous proof (non -aligned implies aligned).

Assume for all $w \in A^\ell$, $\lim_{n \to \infty} \frac{|x[1 \dots n]|_w}{n} = b^{-\ell}$.

Fix ℓ and $w \in A^\ell$. Fix ε. Let k be large enough so that $\widetilde{B}ad = \widetilde{B}ad(A, k\ell, w, \varepsilon)$ has cardinality less than εb^k.

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\|x[1\dots n\ell]\|_w}{n} \leq\ & \limsup_{n\to\infty} \frac{1}{n} \frac{1}{k\ell-\ell+1} \sum_{t=1}^{n\ell-k\ell} \|x[t\dots t+k\ell)\|_{w,*} \\
=& \limsup_{n\to\infty} \frac{1}{n} \frac{1}{k\ell-\ell+1} \sum_{v\in A^{k\ell}} |x[1\dots n\ell]|_v \ \|v\|_{w,*} \\
\leq & \sum_{v\in A^{k\ell}} \left(\limsup_{n\to\infty} \frac{|x[1\dots n\ell]|_v}{n\ell} \right) \frac{\|v\|_{w,*}}{k-1} \\
=& \sum_{v\in A^{k\ell}} b^{-k\ell} \frac{\|v\|_{w,*}}{k-1} \\
=& \sum_{v\in A^{k\ell}\setminus\widetilde{B}ad} b^{-k\ell} \frac{\|v\|_{w,*}}{k-1} + \sum_{v\in \widetilde{B}ad} b^{-k\ell} \frac{\|v\|_{w,*}}{k-1} \\
\leq & b^{k\ell} b^{-k\ell} \frac{k\ell b^{-\ell} + \varepsilon k\ell}{\ell(k-1)} + \varepsilon b^{k\ell} b^{-k\ell} \frac{k\ell}{\ell(k-1)}. \\
=& b^{-\ell}(1+\varepsilon b^\ell) \frac{k}{k-1} + \varepsilon \frac{k}{k-1}
\end{aligned}
$$

We obtained

$$\limsup_{n\to\infty} \frac{\|x[1\dots n\ell]\|_w}{n} \le b^{-\ell}(1+\varepsilon b^{\ell})\frac{k}{k-1} + \varepsilon\frac{k}{k-1}.$$

This inequality holds for every ε and every k large enough, we have

$$\limsup_{n\to\infty} \frac{\|x[1\dots n\ell]\|_w}{n} \le b^{-\ell}.$$

Since this holds for every $w \in A^{\ell}$, by Lemma 4,

$$\lim_{n\to\infty} \frac{\|x[1\dots n\ell]\|_w}{n} = b^{-\ell}.$$

3. We prove that *Aligned normality* implies *Strong aligned normality*.

It is sufficient to prove that if x has aligned normality then bx also has aligned normality. Define

$$U(k, w, i) = \{u \in A^k \ : \ u[i \ldots i + |w| - 1] = w\}.$$

Fix a positive integer ℓ. For any $w \in A^\ell$ and for any positive integer r,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\|(bx)[1 \ldots nr\ell]\|_w}{nr} &\geq \liminf_{n\to\infty} \frac{1}{r} \sum_{k=0}^{r-2} \sum_{u\in U(r\ell,w,2+\ell k)} \frac{\|x[1 \ldots nr\ell]\|_u}{n} \\
&= \frac{1}{r} \sum_{k=0}^{r-2} \sum_{u\in U(\ell r,w,2+\ell k)} b^{-r\ell} \\
&= \frac{r-1}{r} b^{-\ell}.
\end{aligned}$$

For every r the following equality holds:

$$\liminf_{n\to\infty} \frac{\|(bx)[1 \ldots n\ell]\|_w}{n} = \liminf_{n\to\infty} \frac{\|(bx)[1 \ldots nr\ell]\|_w}{nr}.$$

Then, using the inequality obtained above we have,

$$\liminf_{n\to\infty} \frac{\|(bx)[1\ldots n\ell]\|_w}{n} \geq \frac{r-1}{r} b^{-\ell}.$$

Since this last inequality holds for evey r, we obtain,

$$\liminf_{n\to\infty} \frac{\|(bx)[1\ldots n\ell]\|_w}{n} \geq b^{-\ell}.$$

Finally, this last inequality is true for every $w \in A^\ell$, hence by Lemma 4,

$$\lim_{n\to\infty} \frac{\|(bx)[1\ldots n\ell]\|_w}{n} = b^{-\ell}.$$