Departamento de Computación, Facultad de Ciencias Exactas y Naturales, UBA

Azar y Autómatas

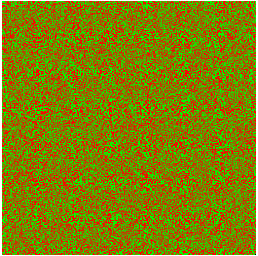
Clase 4: Construcción de números absolutamente normales - Normalidad como d.u. módulo 1

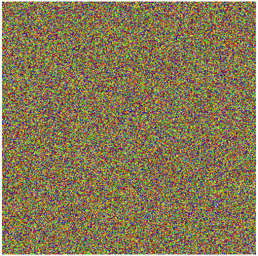
Normal to a given base

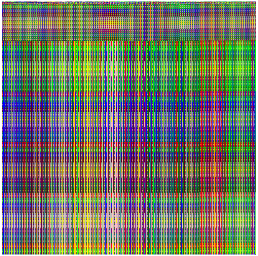
Theorem (Champernowne, 1933)

0.123456789101112131415161718192021222324 . . . *is normal to base* 10*.*

It is unknown if it is normal to bases that are not powers of 10.


base 2


base 6


base 10

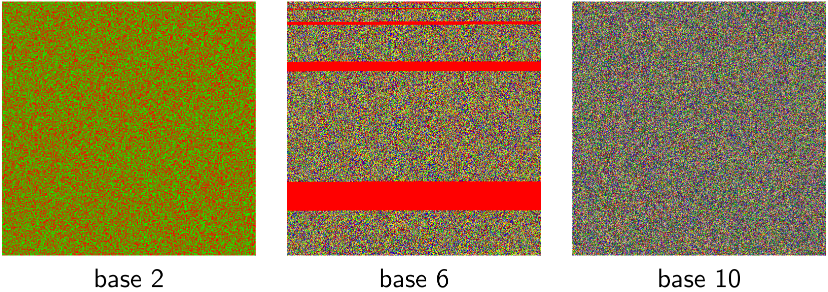
Plots of the first 250000 digits of Champernowne's number.

Normal to one base, but not to another

Theorem (Stoneham, 1973)

$$\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \; 2^{3^k}}$$

is normal to base 2 *but* *not simply normal to base* 6.



Plots of the first 250000 digits of Stoneham number $\alpha_{2,3}$.

Normal to all bases

A real number x is absolutely normal if x is normal to all bases.

Theorem (Borel 1909)

The set of absolutely normal numbers in $[0,1]$ *has Lebesgue measure* 1.

Problem (Borel 1909)

Give one example.

Problem (Borel, 1909)

Are the usual mathematical constants, such as π, *e, or* $\sqrt{2}$, *absolutely normal? Or at least simply normal to* *some* *base?*

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Normal to all bases

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q [1] tout nombre réel x dont la partie fractionnaire

[1] E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

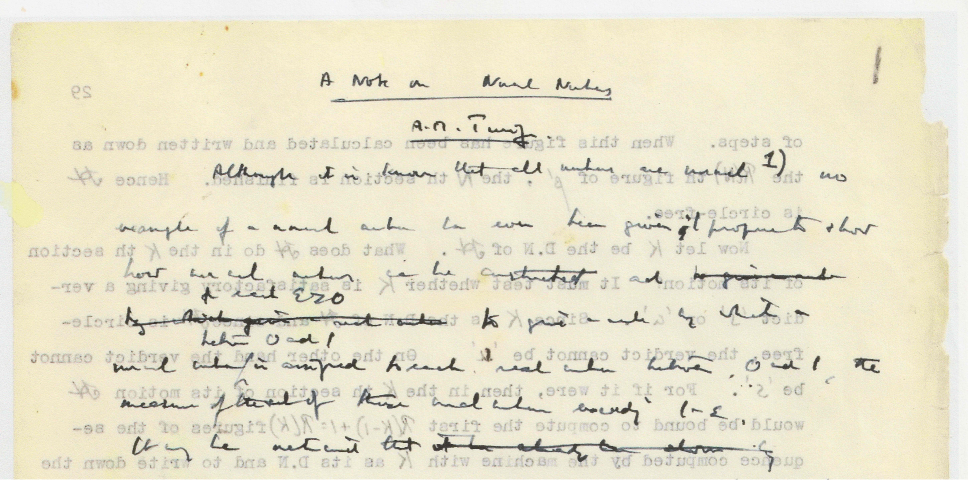
SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Normal to all bases

Turing, A. M. A Note on Normal Numbers. Collected Works of Alan M. Turing, Pure Mathematics, 117-119. Notes of editor J.L. Britton, 263-265. North Holland, 1992.



A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

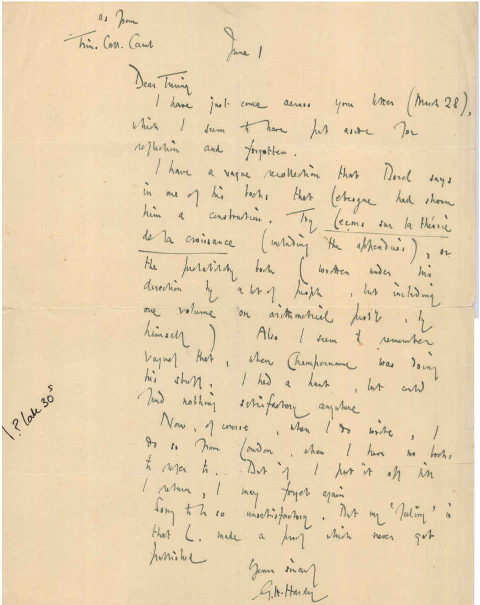
Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t, \gamma, n, R)$ the number of these in which γ occurs exactly n times. Then it can be ~~pr~~oved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t, \gamma, n, R)}{\sum_{n=1}^{R} N(t, \gamma, n, R)} = \frac{R - r + 1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible[2] to prove that

Corrected and completed in Becher, Figueira and Picchi, 2007.

Letter exchange between Turing and Hardy (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob, by himself). Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

? late 30s

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2. *there is a computable function $f : \mathbb{N} \to \{0, 1\}$ such that $f(n)$ is the n-th digit in the fractional expansion of x in base 2.*

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3. *there is a computable non-decreasing sequence of rational numbers $(q_j)_{j \geq 1}$ such that $\lim_{j \to \infty} q_j = x$ and for each j, $|x - q_j| \leq 2^{-j}$.*

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4. *there is a computable sequence of intervals $I_1, I_2, I_3 \ldots$ with rational endpoints nested, whose lengths go to 0 such hat $x \in \bigcap_{j\geq 1} I_j$*

Examples: 0, $\sqrt{2}$, π, e.
Counterexample: Cantor's diagonal argument.

Constructions of normal numbers

Concatenation works if we consider just one base.
For two bases, concatenation in general fails.

For example,

	base 10	*base* 3
$x =$	$(0.25)_{10} =$	$(0.020202020202\ldots)_3$...
$y =$	$(0.0017)_{10} =$	$(0.000001020110110010 2\ldots)_3$
$x + y =$	$(0.2517)_{10} =$	$(0.0202101110122\ldots)_3$

Turing's construction of absolutely normal numbers

Theorem 1 (Turing 1937?)

There is an algorithm that computes the expansion in base 2 of an absolutely normal number in the unit interval.

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This algorithm has double exponential computational complexity:
to compute the first n-th digits in the expansion in base 2 the algorithm perfoms double exponential in n mathematical operations.

Turing's absolutely normal number

Turing uses dyadic intervals. To select $I_1, I_2, I_3 \dots$ his strategy is to "follow the measure". The computed number x is the trace of left/right choices.

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For the construction, the most convenient definition of absolute normality is:

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We define

$$\Delta_N(x_b) = \max_{d \in \{0,\ldots,b-1\}} \left| \frac{|x_b[1 \ldots N]|_d}{N} - \frac{1}{b} \right|$$

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Then, x is simply normal to base b if

$$\lim_{N\to\infty} \Delta_N(x_b) = 0$$

The construction is done by steps

We use n as the step number and define the following functions of n:

$$N_n \quad = \quad 2^{n_0+2n}, \text{where } n_0 = 11, \text{ the number of digits looked at step } n$$

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Observe that b_n is greater than or equal to 2 non-decreasing and unbounded; N_n is non-decreasing and unbounded; ε_n is non-increasing and goes to zero.

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The value n_0 is just to make the forthcoming calculations simple.

Turing's sets of candidates

Define the following sets of real numbers,

$$E_0 = (0,1), \text{ and for each } n$$
$$E_n = \bigcap_{b \in \{2,..,b_n\}} \{x \in (0,1) : \Delta_{N_n}(x_b) < \varepsilon_n\}.$$

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Turing's algorithm

Initial step, $n = 0$. $I_0 = (0, 1)$, $E_0 = (0, 1)$.

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Recursive step, $n > 1$. In the previous step we computed I_{n-1}.
Let I_n^0 be left half of I_{n-1} and I_n^1 be right half of I_{n-1}.
If $\mu\left(I_n^0 \cap \bigcap_{j=0}^{n} E_j\right) > 1/N_n$ then let $I_n = I_n^0$ and $y_n = 0$.
Else let $I_n = I_n^1$ and $y_n = 1$.

The outputs is $y_1 y_2 y_3 \ldots$

Recall that there are a few bad words

Let A be an alphabet of b symbols. The set of words of length k such that a given digit d has a number of occurrences that differs from the expected $1/b$ in plus or minus εk,

$$Bad(A, k, d, \varepsilon) = \left\{ v \in A^k : \left| \frac{|v|_d}{k} - \frac{1}{b} \right| \geq \varepsilon \right\}.$$

Lemma 2 (Adapted from Hardy and Wright)

Let b be an integer greater than or equal to 2 and let k be a positive integer. If $6/k \leq \varepsilon \leq 1/b$ then for every $d \in A$,

$$|Bad(A, k, d, \varepsilon)| < 4b^k e^{-b\varepsilon^2 k/6}.$$

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Proof. The values of N_n and ε_n satisfy the hypotheses of Lemma 2, so,

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Then, for $N_n > e^{10}$ can be checked that

$$\mu((0,1) \setminus E_n) \leq \sum_{b=2}^{b_n} 2b^2 e^{-\varepsilon^2 bN_n/6} < 1/N_n^2. \qquad \text{Hence,}$$

$$\mu E_n \geq 1 - \sum_{b=2}^{b_n} 2b^2 e^{-\varepsilon^2 bN_n/6} \geq 1 - 1/N_n^2. \qquad \square$$

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\frac{|x_b[1\ldots M]|_d}{M} &\geq \frac{|x_b[1\ldots N_n]|_d}{N_{n+1}} &\geq \frac{N_n}{N_{n+1}}\left(\frac{1}{b}-\varepsilon_n\right) = \frac{1}{4}\left(\frac{1}{b}-\varepsilon_n\right).
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Since ε_n is decreasing in n and goes to 0, for each base $b = 2, 3\ldots$,

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\end{aligned}$$

Since ε_n is decreasing in n and goes to 0, for each base $b = 2, 3\ldots$,

$$\limsup_{N\to\infty} \frac{|x_b[1\ldots N]|_d}{N} < 4\frac{1}{b}.$$

Proof of Lemma 1 continuation

Then, for each base b, and every digit in base b,

$$\limsup_{N\to\infty} \frac{|x_b[1\ldots N]|_d}{N} < 4\frac{1}{b^\ell}.$$

By Piatetski-Shapiro Theorem, x is simply normal to every base b.
Hence , x is absolutely normal.$\square$

Proof of Theorem 1

From the previous algorithm follows that

$$0.y_1y_2y_3\ldots \in \bigcap_{n\geq 1} I_n$$

the intervals $I_1, I_2, \ldots$ are nested and for each n, $\mu I_n = 1/2^n$.

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To prove the correctness of the algorithm we need to prove that the following condition is invariant along every step n of the algorithm:

$$\mu\left(I_n \cap \bigcap_{j=1}^{n} E_j\right) > 0.$$

We prove it by induction on n. Recall $N_n = 2^{n_0+2n}$.
Base case $n = 0$.

$$\mu(I_0 \cap E_0) = \mu((0,1)) = 1 > \frac{1}{N_0^2} = \frac{1}{2^{2n_0}}.$$

Proof of Theorem 1

Inductive case, $n > 0$. Assume as inductive hypothesis that

$$\mu\left(I_n \cap \bigcap_{j=0}^{n} E_j\right) > \frac{1}{N_n}.$$

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$$\mu\left(I_n \cap \bigcap_{j=0}^{n+1} E_j\right) = \mu\left(\left(I_n \cap \bigcap_{j=0}^{n} E_j\right) \cap E_{n+1}\right) > \frac{1}{N_n} - \frac{1}{N_{n+1}^2} > \frac{2}{N_{n+1}}.$$

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Since the algorithm chooses I_{n+1} among I_n^0 and I_n^1 ensuring $\mu(I_{n+1} \cap \bigcap_{j=0}^{n+1} E_j) > 1/N_{n+1}$, we conclude $\mu(I_{n+1} \cap \bigcap_{j=0}^{n+1} E_j) > 1/N_{n+1}$ as required.

Proof of Theorem 1

Finally, since $(I_n)_{n\geq 0}$ is nested and $\mu(I_n \cap \bigcap_{j=0}^{n} E_j) > 0$, for every n,

$$\bigcap_{n\geq 0} I_n = \bigcap_{n\geq 0} \left(I_n \cap \bigcap_{j=0}^{n} E_j \right).$$

By Lemma 1, all the elements in $\bigcap_{j\geq 0} E_j$ are absolutely normal.
Hence the unique point $0.y_1 y_2 \ldots$ in $\bigcap_{n\geq 1} I_n$ is absolutely normal. $\square$

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Proposition 2

Turing's algorithm has double exponential time-complexity.

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We do not count how many elementary operations are implied by each of the mathematical operations, so we neglect the computational cost of performing arithmetical operations with arbitrary precision.

Proof of Proposition 2

At step n the algorithm computes the set $I_{n-1} \cap E_n$.
To do this, first it computes

$$I_{n-1} \cap E_n = \bigcap_{b \in \{2,..,b_n\}} \{x \in I_{n-1} \cap E_{n-1} : \Delta_{N_n}(x_b) < \varepsilon_n\}$$

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To compute $I_n \cap E_n$ it must examine the folllowing number of words

$$(b_n)^{N_n - N_{n-1} - (n-1)}.$$

Since $N_n = 2^{n_0+2n}$ and $b_n = \lfloor \log N_n \rfloor$, this is

$$O\big((2n)^{2^{2n}}\big).$$

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We conclude by noticing that using the set $I_n \cap E_n$ at step n the algorithm determines the $n - th$ binary digit of the computed number. $\square$

Constructions of normal numbers

Based on discrete counting

1917 Absolutely normal. Not computable

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Becher and Yuhjtman (polynomial); Scheerer (doubly exponential)

2017 Absolutely normal. Faster convergence to normality than almost all numbers.Exponential complexity.

Aistleitner, Becher, Scheerer and Slaman

2021 A number x and $1/x$ absolutely normal and continued fraction normal

Becher and Madrischt

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2015 Absolutely normal and Liouville.

Becher, Heiber and Slaman

2019 Normal to all bases except 3

Aistleitner, Becher, Carton

Uniform distribution modulo 1

Let $(x_j)_{j\geq 1}$ be a sequence of real numbers in the unit interval.
The discrepancy of the N first elements is

$$D_N((x_j)_{j\geq 1}) = \sup_{0\leq u<v\leq 1} \left| \frac{|\{j : 1 \leq j \leq N \text{ and } u \leq x_j \leq v\}|}{N} - (v-u) \right|.$$

The sequence $(x_j)_{j\geq 1}$ is uniformly distributed in the unit interval if

$$\lim_{N\to\infty} D_N((x_j)_{j\geq 1}) = 0.$$

Theorem 2 (see Kuipers and Niederreiter 2006)

Almost all sequences of real numbers are uniformly distributed modulo 1*.*

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Schmidt (1972) proved that for every sequence $(x_j)_{j\geq 1}$ of reals in the unit interval there are infinitely many Ns such that

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There are sequences that achive this lower bound (see Drmota and Tichy 1997)

Normality in terms of uniform distribution modulo 1

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A real number x is normal to base b if and only if $(b^j x)_{j \geq 0}$ is uniformly distributed modulo 1.

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By way of Weyl's criterion of equidistribution: x is normal to base b if and only if for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i t b^j x} = 0.$$

Suggestion: Try it in Wolframalpha with $x = \pi$: $x = e$: $x = 1/2$, etc.

Speed of convergence to normality

The discrepancy modulo 1 of the sequence $(b^j x)_{j \geq 0}$ gives the speed of convergence to normality to base b.

Theorem 4 (Gál and Gál 1964: Philipp 1975: Fukuyama 2008)

For almost all real numbers x the discrepancy modulo 1 of the sequence $(b^j x)_{j \geq 0}$ is essentially the same and it obeys the law of iterated logarithm up to a constant factor that depends on b.

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For every real $\theta > 1$, there is a constant C_θ such that for almost all real numbers x there is $N_{\theta,0}$ such that for every $N \geq N_{\theta,0}$,

$$D_N((\theta^j x \mod 1)_{j\geq 0}) \leq C_\theta \sqrt{\frac{\log\log N}{N}}.$$

For instance, in case θ is an integer greater than or equal to 2,

$$C_\theta = \begin{cases} \sqrt{84}/9, & \text{if } \theta = 2 \\ \sqrt{2(\theta+1)/(\theta-1)}/2, & \text{if } \theta \text{ is odd} \\ \sqrt{2(\theta+1)\theta(\theta-2)/(\theta-1)^3}/2, & \text{if } \theta \geq 4 \text{ is even.} \end{cases}$$

A number that goes faster to absolute normality

Theorem 5 (Aistleitner, Becher, Sheerer and Slaman 2017)

There is an absolutely normal number x such that for each integer $b \geq 2$, there are numbers $N_0(b)$ and C_b such that for all $N \geq N_0(b)$,

$$D_N((b^j x \mod 1)_{j \geq 0}) \leq \frac{C_b}{\sqrt{N}}.$$

and we can choose the constant $C_b = 3433\ b$.

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and we can choose the constant $C_b = 3433\ b$.

Moreover, there is an algorithm that computes the first N digits of the expansion of x in base 2 after performing exponential in N mathematical operations.

Much faster to absolute normality

Theorem 5 does not supersede the discrepancy bound obtained by Levin (1999) in one fixed base:

For a fixed integer $b \geq 2$, Levin constructed a real number x such that for every N sufficiently large

$$D_N((b^j x \mod 1)_{j \geq 0}) < \frac{C_\theta (\log N)^2}{N}.$$

Two central questions remain open

1. Asked by Korobov 1955:
For a *fixed* integer $b \geq 2$, what is the function $\psi(N)$ with maximal speed of decrease to zero such that there is a real number x for which

$$D_N((b^j x \mod 1)_{j \geq 0}) = \mathcal{O}(\psi(N)) \qquad \text{as } N \to \infty?$$

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2. Asked by Bugeaud (personal communication, 2017):
Is there a number x satisfying the minimal discrepancy estimate for normality not only in one fixed base, but in all bases at the same time?

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2. Asked by Bugeaud (personal communication, 2017):
Is there a number x satisfying the minimal discrepancy estimate for normality not only in one fixed base, but in all bases at the same time? More precisely, let ψ be Korobov's function from above. Is there a real number x such that for *all* integer bases $b \geq 2$,

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