INSERTION
IN CONSTRUCTED NORMAL NUMBERS

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ABSTRACT. Defined by Borel, a real number is normal to an integer base $b \geq 2$ if in its base-b expansion every block of digits occurs with the same limiting frequency as every other block of the same length. We consider the problem of *insertion* in constructed base-b normal expansions to obtain normality to base $(b + 1)$.

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1. Problem description and statement of results

Defined by Émile Borel, a real number is normal to an integer base $b \geq 2$ if in its base-b expansion every block of digits occurs with the same limiting frequency as every other block of the same length. Equivalently, a real number x is normal to base b if the fractional parts of $x, bx, b^2x, \ldots$ are uniformly distributed modulo one in the unit interval.

There are many ways to modify normal numbers preserving normality to a given base. A major result is Wall's theorem [10] showing that the subsequences of a base-b expansion along arithmetic progressions preserve normality, crowned by Kamae's and Weiss' [14] complete characterization of the subsequences that preserve normality. Other normality preserving operations are addition by some numbers [1, 19, 22], multiplication by a rational [10], transformations by some finite automata [8] and there are more.

Another form of modification transfers normality from base b to normality to base $(b - 1)$: Vandehey [21, Theorem 1.2] proved that, when $b \geq 3$, the subsequence of a base-b normal expansion formed by all the digits different from $(b - 1)$ is normal to base $(b - 1)$. Thus, the removal of all the instances of the digit $(b - 1)$ from a normal base-b expansion yiellds a sequence normal to base b.

Here we consider the dual problem of transferring normality from base b to base $(b + 1)$. Specifically, we consider the following:

PROBLEM. How to *insert* digits along a normal base-b expansion so that the resulting expansion is normal to base $(b + 1)$?

There are two versions of the insertion problem:

  – when insertion freely uses all the digits in base $(b + 1)$,

  – when insertion is limited just to the new digit.

In the present work we tackle the free insertion problem on a class of *constructed* normal numbers. Since we look at normality to just one base at a time, instead of fractional expansions of real numbers we deal with sequences of symbols in a given alphabet and we talk about normality to that alphabet. We state the results as transferring normality from an alphabet A to the alphabet $\widehat{A} = A \cup \{\sigma\}$ with σ not in A.

We consider constructed sequences that are the concatenation of *perfect necklaces* over the alphabet A of linearly increasing order. After insertion, the resulting sequence is also a concatenation of perfect necklaces of linearly increasing order but over the alphabet $\widehat{A}$. Perfect necklaces were introduced in [2]. They are a variant of the classical de Bruijn sequences. The of perfect necklaces of linearly increasing order is a normal sequence (this is proved in Proposition 6).

We give an effective construction that controls the distance between each occurrence of the new digit and the next. An effective construction is a prescription on how to perform the insertion while reading the input sequence from left to right. We prove the following.

THEOREM 1. *Let A and $\widehat{A}$ be alphabets such that $\widehat{A} = A \cup \{\sigma\}$ with σ not in A. Let $v \in A^{\omega}$ be the concatenation of (n, n)-perfect necklaces over the alphabet A for $n = 1, 2, \ldots$ Then, there is an effective construction of a sequence $\widehat{v} \in \widehat{A}^{\omega}$ normal to the alphabet $\widehat{A}$ such that v is a subsequence of $\widehat{v}$ and $\widehat{v}$ is the concatenation of (n, n)-perfect necklaces over the alphabet $\widehat{A}$ for $n = 1, 2, \ldots$ Moreover, for every integer N greater than $|A|$, in between the occurrences of the symbol σ in $\widehat{v}$ just before and just after position N, there are at most $2|A| + \log_{|\widehat{A}|}(N)$ symbols.*

The one symbol insertion problem has already an adroit solution on arbitrary normal sequences:

THEOREM 2 (Zylber [23, Theorem 1]). *Let A and $\widehat{A}$ be alphabets such that $\widehat{A} = A \cup \{\sigma\}$ with σ not in A. Let the sequence $v \in A^{\omega}$ be normal to the alphabet A. Then, there exists a sequence $\widehat{v} \in \widehat{A}^{\omega}$ normal to the alphabet $\widehat{A}$ such that $r(\widehat{v}) = v$, where r is the retract that removes all the instances of the symbol σ.*

Zylber gives a construction that, in general, is not effective. It becomes effective when the input sequence $v \in A^{\omega}$ satisfies the following condition. First consider the simple discrepancy of a sequence $(x_n)_{n \geq 1}$ of real numbers in the unit interval with respect to an interval I,

$$d_{N,I}\big((x_n)_{n \geq 1}\big) = \left| \frac{1}{N} \#\{n : 1 \leq n \leq N, x_n \in I\} - |I| \right|.$$

Let b be the cardinality of the alphabet A and let x be the real number whose expansion in base b is the input sequence $v \in A^{\omega}$. Zylber's construction becomes effective when there is a computable upper bound of the simple discrepancies

$$d_{N,I}\big((b^{\ell n})x \bmod 1\big)_{n \geq 1}$$

for infinitely many integer values ℓ, and for every interval I of the form

$$\left(\frac{a}{b^{\ell}}, \frac{a+1}{b^{\ell}} \right), \quad \text{with} \quad 0 \leq a < b^{\ell} - 1.$$

There are many cases where this condition holds. It follows from Proposition 4 that one instance is the concatenation of the n-ordered necklaces for $n = 1, 2, \ldots$ (the n-ordered necklace is the perfect necklace given by the concatenation of all words of length n in lexicographic order),

It remains to study how to compare the discrepancy of $(b^n x \bmod 1)_{n \geq 0}$ and the discrepancy of $\big((b+1)^n y \bmod 1\big)_{n \geq 0}$ when the base-$(b+1)$ expansion of y results from insertion in the base-b expansion of a normal number x. It may be possible to obtain metric results similar to those obtained by Fukuyama and Hiroshima [12] for some subsequences of $(b^n x \bmod 1)_{n \geq 0}$.

This document is organized as follows: Section 2 presents the basics of perfect necklaces and Section 3 solves the free insertion problem on the concatenation of perfect necklaces.

2. Perfect necklaces and nested perfect necklaces

We start with some notation. A word is a finite sequence of symbols in a given alphabet. For a finite alphabet A, we write $|A|$ for its cardinality, A^n for the set of all words of length n, A^* for the set of all words and A^ω for the set of all infinite sequences.

The positions in words and in sequences are numbered starting at 1. We write $v[i]$ for the symbol at position i and we write $v[i, j]$ for the symbols of v from position i to position j. The length of a word v is $|v|$. Let $\theta : A^* \to A^*$ be the *rotation* operator,
$$(\theta v)[i] = v[(i + 1) \bmod |v|].$$

We write θ^n for the application of the rotation n times.

A circular word or necklace is the equivalence class of a word under rotations. To denote a necklace we write $[w]$, where w is any of the words in the equivalence class. E.g., $[000]$ contains a single word 000 because for every i, $\theta^i(000) = 000$, and $[110]$ contains three words because
$$\theta^0(110) = 110, \quad \theta^1(110) = 101 \quad \text{and} \quad \theta^2(110) = 011.$$

2.1. Perfect necklaces

Perfect necklaces are a variant of the classical de Bruijn sequences and they were introduced in [2]. A de Bruijn sequence of order n over the alphabet A is a necklace of length $|A|^n$ and each word of length n occurs in it exactly once.

Definition (Perfect necklace). A necklace is (n, k)-perfect if each word of length n occurs k many times at positions different modulo k, for any convention of the starting point.

Thus, each (n, k)-perfect necklace has length $k|A|^n$. Notice that $(n, 1)$-perfect necklaces coincide with the de Bruijn sequences of order n.

Consider the alphabet $A = \{0, 1\}$. The following are $(2, 2)$-perfect necklaces,
$$[00\ 01\ 10\ 11] \quad \text{and} \quad [00\ 10\ 01\ 11].$$

This is a $(3, 3)$-perfect necklace
$$[000\ 110\ 101\ 111\ 001\ 010\ 011\ 100].$$

The following are not (n, n)-perfect

$$[00\ 01\ 11\ 10] \quad \text{and} \quad [000\ 101\ 110\ 111\ 010\ 001\ 011\ 100].$$

DEFINITION (Ordered necklace). For an alphabet A and a positive integer n, the *n-ordered necklace* is the concatenation of all words of length n in lexicographic order.

These are the n-ordered necklaces over the alphabet $A = \{0, 1\}$ for $n = 1, 2, 3$,

$$[01], \quad [00\ 01\ 10\ 11], \quad [000\ 001\ 010\ 011\ 100\ 101\ 110\ 111].$$

Every n-ordered necklace is (n, n)-perfect. Inexplicably, this was not observed by Barbier [3,4] nor by Champernowne [9].

REMARK ([2, Theorem 5]). *Identify words of length n over the alphabet A with the integers 0 to $|A|^n - 1$. Let r coprime with $|A|$. The concatenation of words corresponding to the arithmetic sequence $0, r, 2r, \ldots, (|A|^n - 1)r$ yields a (n, n)-perfect necklace. By taking $r = 1$ we obtain that the n-ordered necklaces are (n, n)-perfect.*

PROPOSITION 1. *In the n-ordered necklace over the alphabet A, for each symbol $a \in A$, in between two occurrences of a there can be up to $n|A| - 1$ symbols.*

P r o o f. The n-ordered necklace is the concatenation of all words of length n in lexicographical order. Consider $|A| + 1$ many consecutive of these words, $u_1, \ldots u_{|A|+1}$. Observe that the last symbol in u_1 is necessarily the same as the last symbol in $u_{|A|+1}$. Let a be that symbol. In between these two occurrences of a there are $n|A| - 1$ symbols. For some choices of $u_1, \ldots u_{|A|+1}$ these are the only two occurrences of a in these words. All the other cases yield a smaller number of symbols between two occurrences of a. □

A particular class of perfect necklaces, called *nested perfect necklaces*, were introduced in [5] generalizing a construction given by M. Levin in [16, Theorem 2]. A (n, k)-perfect necklace over the alphabet A is *nested* if $n = 1$ or it is the concatenation of $|A|$ nested $(n - 1, k)$-perfect necklaces. For example, the following is a nested $(2, 2)$-perfect necklace over the alphabet $A = \{0, 1\}$,

$$\left[\underbrace{0011}_{(1,2)\text{-perfect}} \underbrace{0110}_{(1,2)\text{-perfect}} \right].$$

Each of these 8 are $(1, 4)$-perfect necklaces.

$$
\begin{array}{ll}
[00001111], & [01011010], \\
[00111100], & [01101001], \\
[00011110], & [01001011], \\
[00101101], & [01111000].
\end{array}
$$

- The concatenation in each row yields a $(2, 4)$-perfect necklace.
- The concatenation of the first two rows yields a nested $(3, 4)$-perfect necklace.
- The concatenation of the last two rows yields a nested $(3, 4)$-perfect necklace.
- The concatenation of all rows yields a nested $(4, 4)$-perfect necklace.

The n-ordered necklaces are perfect but not nested, e.g., for

$$
A = \{0, 1, \sigma\} \quad \text{and} \quad n = 2,
$$

$$
\left[
\underbrace{00\ 01\ 0\sigma}_{\text{not } (1,2)\text{-perfect}}
\quad
\underbrace{10\ 11\ 1\sigma}_{\text{not } (1,2)\text{-perfect}}
\quad
\underbrace{\sigma 0\ \sigma 1\ \sigma\sigma}_{\text{not } (1,2)\text{-perfect}}
\right].
$$

2.2. Perfect necklaces as Eulerian cycles in astute graphs

The (n, k)-perfect necklaces are characterized with Eulerian cycles in the so called *astute graphs*.

DEFINITION (Astute graph)**.** The astute graph $G_A(n, k)$ is a pair (V, E) where

$$
V \quad = \quad \big\{(w, m) : w \in A^n,\ m \in \{0, \ldots k - 1\}\big\}
$$
and
$$
E \quad = \quad \big\{\big((w, m), (w', m')\big) : w[2, n] = w'[1, n - 1],\ m' = (m + 1) \bmod k\big\}.
$$

Thus, $G_A(n, k)$ has $k|A|^n$ vertices and $k|A|^{n+1}$ edges. It is Eulerian because it is strongly regular (all vertices have in-degree and out-degree equal to $|A|$) and strongly connected (every vertex is reachable from every other vertex). Notice that $G_A(n, 1)$ is the de Bruijn graph of words of length n over the alphabet A.

PROPOSITION 2 ([2, Corollary 14])**.** *Each (n, k)-perfect necklace over the alphabet A can be constructed as an Eulerian cycle in $G_A(n - 1, k)$.*

In some cases several Eulerian cycles in $G_A(n - 1, k)$ yield the same (n, k)-perfect necklace, this happens when there is a period inside a cycle.

REMARK ([2, Theorem 20]). *The number of (n, k)-perfect necklaces over a b- -symbol alphabet is*

$$\frac{1}{k} \sum_{d_{b,k}|j|k} e(j)\phi(k/j),$$

where:

- $d_{b,k} = \prod p_i^{\alpha_i}$, *such that $\{p_i\}$ is the set of primes that divide both b and k, and α_i is the exponent of p_i in the factorization of k,*
- $e(j) = (b!)^{jb^{n-1}}b^{-n}$ *is the number of Eulerian cycles in $G_A(n-1, j)$, where $|A| = b$,*
- ϕ *is Euler's totient function, $\phi(m)$ counts the positive integers less than or equal to m that are relatively prime to m.*

REMARK ([5, Theorem 2]). *For each $d = 0, 1, 2, \dots$ there are $2^{2^{d+1}-1}$ binary nested $(2^d, 2^d)$-perfect necklaces.*

2.3. Counting aligned and non-aligned occurrences of words

For the number of occurrences of a word u in a word v at any position we write $|v|_u$,

$$|v|_u = \big|\{i : v[i, i + |u| - 1] = u\}\big|.$$

For example, $|00010|_{00} = 2$. We are interested in counting occurrences of a word u in a word v when $[v]$ is a perfect necklace.

PROPOSITION 3. *If $[v]$ is a (n, k)-perfect necklace over the alphabet A, then for every word u of length at most n,*

$$k|A|^{|v|-|u|} - |u| + 1 \le |v|_u \le k|A|^{|v|-|u|}.$$

Recall that the positions in words are numbered starting at 1. Given two words v and u, we write $\|v\|_u$ for the number of occurrences of u at the positions of v congruent to 1 modulo the length of u, that we call *aligned occurrences*,

$$\|v\|_u = \big|\{i : v[i, i + |u| - 1] = u \quad \text{and} \quad i \equiv 1 \bmod |u|\}\big|.$$

For example,

$$\|00000\|_{00} = 2 \quad \text{and} \quad \|1001\|_{00} = 0.$$

The relation between $|v|_u$ and $\|v\|_u$ is as follows,

$$|v|_u = \sum_{i=0}^{|u|-1} \|v[1+i, |v|]\|_u.$$

So, for any single symbol a in the alphabet A, $|v|_a = \|v\|_a$.

Consider the alphabet A with cardinality b. Identify infinite sequences in A^ω with real numbers in the unit interval according to their expansion in base b. Identify the ith word in the lexicographic order among all the words in A^ℓ with the ith interval among the b^ℓ intervals

$$\left(\frac{a}{b^\ell}, \frac{a+1}{b^\ell}\right) \quad \text{for} \quad 0 \le a < b^\ell.$$

Then, the number of aligned occurrences of a word u in A^ℓ in initial segments of a sequence $v \in A^\omega$ allows us to compute the simple discrepancy

$$d_{N,I}\big((b^{\ell n}v \bmod 1)_{n \ge 1}\big) = \left| \frac{1}{N} \#\{n : 1 \le n \le N, \big(b^{\ell n}v \bmod 1\big) \in I\} - |I| \right|,$$

where I is the interval identified with the word u.

PROPOSITION 4.

(1) *If $[v]$ is (n,n)-perfect over the alphabet A, then for every word u of length ℓ, where ℓ divides n,*

$$\|v\|_u = |A|^{n-\ell} n/\ell.$$

(2) *If $[v]$ is the n-ordered necklace, then for every word u of length ℓ, where ℓ divides n, and for any position t in v,*

$$|A|^{-\ell}t/\ell - O(t/n) \le \|v[1,t]\|_u \le |A|^{-\ell}t/\ell + O(t/n).$$

P r o o f.

(1) To count the number of occurrences of u of length ℓ in $[v]$, with $1 \le \ell \le n$, we count how many times u occurs at the beginning of a word of length n. There are $|A|^{n-\ell}$ many different words of length n that start with u, and each occurs n times in $[v]$ at positions that are different modulo n. Notice that v has length $n|A|^n$, which is multiple of n and multiple of ℓ.

In the last n positions of v there are n/ℓ of them that are multiple of ℓ, and each of them can be the start of a circular occurrence of some uw with $|uw| = n$. Thus, number of aligned occurrences of u in v is

$$|A|^{n-L}n/\ell.$$

and coincides with the number of aligned occurrences of u in the necklace $[v]$.

(2) Suppose v is the concatenation of blocks $B_0 \ldots B_{|A|^n - 1}$, where each block has the form

$$q_{n-1} q_{n-2} \cdots q_{-1} q_0,$$

the qs are symbols in A. Identify the symbols in A with the digits in bas $|A|$. Suppose the tth symbol of v occurs within $p_{n-1} p_{n-2} \ldots p_1 p_0$ then

$$t = n \sum_{j=0}^{n-1} p_j |A|^j + \theta n, \quad \text{with} \quad 0 < \theta \leq 1.$$

Suppose u has length ℓ less than or equal to n. Let $g_{n,k}(v, t, u)$ denote the number of times that u occurs undivided in the first t digits of v with the first digit of u as the kth digit of a block in v. Let us say that a block is of type B if the word u occurs at position k. We count the number of blocks of type B in v up to position t. With u fixed in position k in the blocks of type B,

$$\underbrace{\overbrace{p_{n-1} p_{n-2} \cdots p_{n-(k-1)}}^{\text{first } k-1 \text{ digits of the block}} \quad \overbrace{u_1 u_2 \ldots u_\ell}^{\text{digits of } u} \quad \overbrace{\cdots}^{n-k-\ell+1 \text{ digits}}}_{\text{block } B}$$

we may choose the last $n - \ell - k + 1$ digits of the blocks in $|A|^{n-\ell-k+1}$ ways. Having chosen these, in order to ensure that the blocks of type B lay in $v[1, t]$, we shall be able to choose the first $k - 1$ digits of the blocks of type B in this number of ways,

$$\sum_{j=n-(k-1)}^{n-1} p_j |A|^{j+k-n-1}$$

or

$$\sum_{j=n-(k-1)}^{n-1} p_j |A|^{j+k-n-1} + 1.$$

Then, if $k \leq n - \ell + 1$,

$$g_{n,k}(v, t, u) = |A|^{n-\ell-k+1} \sum_{j=n-k+1}^{n-1} p_j |A|^{j+k-n-1} + \theta',$$

$$= |A|^{-\ell} \sum_{j=n-k+1}^{n-1} p_j |A|^j + \theta' |A|^{n-k+1},$$

where $0 \leq \theta' \leq 1$.

Let $K = n \bmod \ell$. To obtain a lower bound of $\|v[1,t]\|_u$ we sum $g_{n,k}(v,t,u)$ for every k such that $(k \bmod \ell) = K$,

$$\|v[1,t]\|_u \geq \sum_{\substack{k:(k \bmod \ell)=K \\ 1 \leq k \leq n-\ell+1}} g_{n,k}(v,t,u)$$

$$\geq \sum_{\substack{k:(k \bmod \ell)=K \\ 1 \leq k \leq n-\ell+1}} |A|^{-\ell} \sum_{j=n-k+1}^{n-1} p_j |A|^j$$

$$= |A|^{-\ell} \sum_{j=\ell+K}^{n-1} \lfloor j/\ell \rfloor p_j |A|^j$$

$$\geq |A|^{-\ell}/\ell \left(\sum_{j=\ell+K}^{n-1} n p_j |A|^j - \sum_{j=\ell+K}^{n-1} (n-j) p_j |A|^j \right)$$

$$= |A|^{-\ell} t/\ell - O(t/n). \qquad \square$$

Finally, notice that the aligned occurrences of u can occur at most once in between every two blocks of v,

$$\|v[1,t]\|_u \leq \left(\sum_{\substack{k:(k \bmod \ell)=K \\ 1 \leq k \leq n-\ell+1}} g_{n,k}(v,t) \right) + O(t/n)$$

$$= \left(\sum_{\substack{k:(k \bmod \ell)=K \\ 1 \leq k \leq n-\ell+1}} |A|^{-\ell} \sum_{j=n-k+1}^{n-1} p_j |A|^j \right) + O(t/n)$$

$$= |A|^{-\ell} \left(\sum_{j=\ell+\eta}^{n-1} \lceil j/\ell \rceil p_j |A|^j \right) + O(t/n)$$

$$\leq |A|^{-\ell}/\ell \left(\sum_{j=\ell+K}^{n-1} n p_j |A|^j - \sum_{j=\ell+\eta}^{n-1} (n-j) p_j |A|^j \right) + O(t/n)$$

$$= |A|^{-\ell} t/\ell + O(t/n).$$

2.4. From perfect necklaces to normal sequences

We show that the concatenation of perfect necklaces of linearly increasing order is normal. To prove it we use Piatetski-Shapiro's theorem [7, 17, 18].

PROPOSITION 5 (Piatetski-Shapiro theorem). *The sequence $v \in A^\omega$ is normal if and only if there is positive constant C such that for all words u,*

$$\limsup_{n \to \infty} \frac{|v[1, n]|_u}{n} \le C|A|^{-|u|}.$$

PROPOSITION 6. *The concatenation of (n, k)-perfect necklaces over the alphabet A, for $n = 1, 2, \dots$ and k_n a linear function of n, is normal to the alphabet A.*

P r o o f. Let $M(0) = 0$ and for $j \ge 1$,

$$M(j) = \sum_{i=1}^{j} k_i |A|^i.$$

Fix N and let m be the minimum integer such that $N \le M(m)$. By Proposition 3 for every u of length ℓ,

$$|v[1, N]|_u \le |v[1, M(m)]|_u$$

$$\le \sum_{i=1}^{m} |v[M(i-1) + 1, M(i)]|_u + \ell - 1$$

$$\le \sum_{i=1}^{m} k_i |A|^{i-\ell} + \ell - 1 \le k_m |A|^{m+1} |A|^{-\ell} + m\ell.$$

Since k_n is linear in n, for every n we have k_n/k_{n-1} is a constant c. Then, using

$$k_{m-1}|A|^{m-1} < M(m-1) \quad \text{and} \quad M(m) \le k_m|A|^{m+1},$$

$$\limsup_{N \to \infty} \frac{|v[1, N]|_u}{N} \le \limsup_{m \to \infty} \frac{|v[1, M(m)]|_u}{M(m-1)}$$

$$\le \limsup_{m \to \infty} \frac{k_m|A|^{m+1}|A|^{-\ell} + m\ell}{k_{m-1}|A|^{m-1}} \le c|A|^2|A|^{-\ell}.$$

Piatetski-Shapiro theorem (Proposition 5) holds with $C = c|A|^2$ and v is normal.
□

Actually Proposition 6 holds for k_n being polynomial in n and the same proof applies. The concatenation of nested perfect necklaces of exponentially increasing order also yields normal sequences, but the same proof does not apply.

Normal numbers are exactly those real numbers x for which $(b^n x)_{n \geq 1}$ is uniformly distributed modulo one (see [7, 11, 15]), which means that the discrepancy of the first N terms

$$D_N\big((b^n x \bmod 1)_{n \geq 0}\big) = \sup_{\gamma \in [0,1)} \left| \frac{1}{N} |\{n \leq N : (b^n x \bmod 1)\}| < \gamma| - \gamma \right|$$

goes to 0 as N goes to infinity. For sequences of the form $(b^n x \bmod 1)_{n \geq 1}$ the smallest known discrepancy of the first N terms is $O\big((\log N)^2/N\big)$, see [7, 16]. Expansions made of nested perfect necklaces of exponentially increasing order yield real numbers x with this property.

REMARK ([5, Theorem 1]). *Let b a prime number. The base-b expansion of the number defined by M.Levin using Pascal triangle matrix modulo 2 is the concatenation of nested $(2^d, 2^d)$-perfect necklaces for $d = 0, 1, 2, \dots$ And for every number x whose base-b expansion is the concatenation of nested $(2^d, 2^d)$-perfect necklaces for $d = 0, 1, 2 \dots$, $D_N\big((b^n x \bmod 1)_{n \geq 0}\big)$ is $O\big((\log N)^2/N\big)$.*

In general, the discrepancy associated to the concatenation of (n, k)-perfect necklaces has not been studied. One exception is the discrepancy associated to the concatenation n-ordered necklaces which is exactly the discrepancy associated to Champernowne's sequence [9] proved in [20], see also [7, 11].

REMARK ([20, Theorem 1]). *The number x whose base b expansion is the concatenation of the n-ordered necklaces for $n = 1, 2, \dots$ $D_N\big((b^n x \bmod 1)_{n \geq 0}\big)$ is $O\big(1/(\log N)\big)$.*

3. Free insertion

3.1. Tools to prove Theorem 1

Consider the alphabets A and $\widehat{A} = A \cup \{\sigma\}$ for σ not in A. Since the length and lexicographic order on words over the alphabet A respects the length and lexicographic order on words over $\widehat{A}$, by inserting suitable symbols in suitable positions in each n-ordered necklace over A we obtain each n-ordered necklace over $\widehat{A}$. For example, for $A = \{0, 1\}$ and $\widehat{A} = \{0, 1, \sigma\}$, and writing inserted symbols in boldface,

0 1 00 0110 11 000 001010 011100 101110 111 0000 0001 …

0 1σ 00 01 **0**σ 10 11 **1**σ σ**0** σ**1** $\sigma\sigma$ 000 001 **00**σ 010 011 **01**σ **0**σ**0** **0**σ**1** **0**$\sigma\sigma$ 100 101 **10**σ 110 111 **11**σ **1**σ**0** **1**σ**1** **1**$\sigma\sigma$ σ**00** σ**01** σ**0**σ σ**10** σ**11** σ**1**σ $\sigma\sigma$**0** $\sigma\sigma$**1** $\sigma\sigma\sigma$ 0000 0001 …

Much more is true: for any (n, k)-perfect necklace over the alphabet A there is a (n, k)-perfect necklace over the alphabet $\widehat{A}$ such that the first is a subsequence of the second. The reason is that, by Proposition 2 perfect necklaces correspond to Eulerian cycles on astute graphs, the astute graph $G_A(k, n-1)$ is a subgraph of $G_{\widehat{A}}(k, n-1)$, and any cycle in an Eulerian graph can be embedded into a full Eulerian cycle. This can be constructed with Hierholzer's algorithm for joining cycles together to create an Eulerian cycle of a graph. However this method does not guarantee that in the resulting (n, k)-perfect necklace over the alphabet $\widehat{A}$, there will be a small gap between one occurrence of the symbol σ and the next.

The following lemma gives a method to insert symbols in a (n, n)-perfect necklace ensuring a *small gap condition*. The lemma extends the work for de Bruijn sequences in [6, Theorem 1].

LEMMA 1 (Main lemma). *Assume the alphabets A and $\widehat{A} = A \cup \{\sigma\}$ for σ not in A. For every (n, n)-perfect necklace $[v]$ over the alphabet A there is a (n, n)-perfect necklace $[\widehat{v}]$ over the alphabet $\widehat{A}$ such that v is a subsequence of $\widehat{v}$. Moreover, for each such $[v]$ there is $[\widehat{v}]$ satisfying that in between any occurrence of the symbol σ and the next there are at most $n + 2|A| - 2$ other symbols.*

By Proposition 1 the n-ordered necklace over the alphabet $\widehat{A}$ fails the small gap condition required in Lemma 1 (Main lemma). For instance, for $A = \{0, 1\}$, $\widehat{A} = \{0, 1, \sigma\}$ and $n = 2$, there are occurrences of σ with more than $n + 2|A| - 2 = 4$ symbols in between (the inserted symbols are in boldface):

$$[v] = \Big[\ 00\ 01_{\blacktriangle}10\ 11_{\blacktriangle}\Big],$$

$$[\widehat{v}] = \left[\underbrace{00\ 01\ \mathbf{0}\sigma}_{5\ symbols}\ \underbrace{10\ 11\ \mathbf{1}\sigma}_{5\ symbols}\ \sigma\mathbf{0}\ \sigma\mathbf{1}\ \sigma\sigma\right].$$

However, this other insertion satisfies the small gap condition:

$$[\widehat{v}] = \left[\underbrace{0}\sigma\ \underbrace{\mathbf{00}\ 01}_{4\ symbols\ \ 3\ symbols}\ \sigma\underbrace{\mathbf{1}\ 10}\ \sigma\sigma\ \underbrace{\mathbf{01}}_{2\ symbols}\ \sigma\sigma\ \underbrace{\mathbf{11}}_{2+1\ symbols}\right].$$

To prove Lemma 1 (Main lemma) we start with a (n, n)-perfect necklace $[v]$ over the alphabet A, we consider an Eulerian cycle in $G_A(n-1, n)$ that corresponds to $[v]$ and we extend it to an Eulerian cycle in graph $G_{\widehat{A}}(n-1, n)$. The (n, n)-perfect necklace that describes this cycle is the wanted $[\widehat{v}]$.

Since for every pair of positive integers n, k, we have that $G_A(n, k)$ is a subgraph of $G_{\widehat{A}}(n, k)$ the following is well defined.

Definition (Augmenting graph). The augmenting graph $X_{\widehat{A}}(n-1, n)$ is the directed graph (V, E), where

$$V = \widehat{A}^{n-1},$$

$$E = \left\{ \begin{array}{ll} \big((au, m), (ub, (m+1)\bmod n)\big): & u \in \widehat{A}^{n-2}, a, b \in \widehat{A} \\ & \big(|au|_\sigma > 0 \text{ or } |ub|_\sigma > 0\big), \\ & m \in \{0, \ldots, n-1\} \end{array} \right\}.$$

Each vertex in $X_{\widehat{A}}(n-1, n)$ that is also a vertex in $G_A(n-1, n)$ has exactly one incoming edge and exactly one outgoing edge. This outcoming edge is associated to new symbol σ.

We say that two cycles are disjoint if they have no common edges. We prove Lemma 1 (Main lemma) constructing an Eulerian cycle in $G_{\widehat{A}}(n-1, n)$ by joining the given Eulerian cycle in $G_A(n-1, n)$ with disjoint cycles of the augmenting graph $X_{\widehat{A}}(n-1, n)$ that we call *petals*. These petals must exhaust the augmenting graph $X_{\widehat{A}}(n-1, n)$. Recall that θ is the rotation operation on words that shifts one position to the right.

Definition (Necklaces on pairs (u, m)). Assume the alphabet $\widehat{A}$ and a positive integer n. For $u \in \widehat{A}^n$ and m between 0 and $n-1$, the necklace $[(u, m)]$ is:

$$[(u, m)] = \big\{ (u, m), (\theta(u), (m+1)\bmod n), \ldots, (\theta^{n-1}(u), (m+n-1)\bmod n) \big\}.$$

Proposition 7. *The set of edges in $G_{\widehat{A}}(n-1, n)$ can be partitioned in disjoint simple cycles identified by the necklaces of pairs $[(u, m)]$, for $u \in \widehat{A}^n$ and m between 0 and $n-1$.*

P r o o f. Let $u \in \widehat{A}^{n-1}$, let $a \in \widehat{A}$ and let $m \in \{0, \ldots n-1\}$. Consider the elements in $[(ua, m)]$ and the sequence of edges

$$(ua, m) \to \big(\theta(ua), (m+1)\bmod n\big) \to \big(\theta^2(ua), (m+2)\bmod n\big) \to \cdots$$
$$\cdots \to \big(\theta^n(ua), (m+n)\bmod\big).$$

The vertices related by these edges are pairwise different except

$$(ua, m) = \big(\theta^n(ua), (m+n)\bmod n\big).$$

Thus, these n edges form a simple cycle in $G_{\widehat{A}}(n-1, n)$. For each congruence class m, the partition of the set of words of length n in the equivalence classes given by their rotations determines a partition of the set of edges in $G_{\widehat{A}}(n-1, n)$ into disjoint simple cycles. $\qquad\square$

Proposition 7 induces the following.

DEFINITION (Graph of necklaces). Define $C_{\widehat{A}}(n, n)$ as the graph (V, E), where

$$V = \{[(u, m)] : u \in \widehat{A}^n, m = 0, \ldots, n - 1\}$$

$$E = \left\{ (x, y) : \begin{array}{l} \text{there is } (au, m) \in x \text{ and there is } (uc, (m + 1) \bmod n) \in y, \\ \text{for } u \in \widehat{A}^{n-1}, a, c \in \widehat{A} \end{array} \right\}.$$

Define the graph $\widehat{C}_{\widehat{A}}(n, n)$ as the subgraph of $C_{\widehat{A}}(n, n)$ whose vertices contain at least one occurrence of the symbol σ.

A petal for a vertex in $G_A(n - 1, n)$ is a union of disjoint cycles in $X_{\widehat{A}}(n - 1, n)$ that correspond to necklaces of length n that have at least one occurrence of symbol σ. These necklaces are vertices in $\widehat{C}_{\widehat{A}}(n, n)$.

DEFINITION (Petal for vertex in $G_A(n - 1, n)$). A *petal* for vertex (u, m) in $G_A(n - 1, n)$ is a cycle in $X_{\widehat{A}}(n - 1, n)$ induced by a subgraph of $\widehat{C}_{\widehat{A}}(n, n)$ that contains the necklace $[(u\sigma, m)]$.

To exhaust $X_{\widehat{A}}(n - 1, n)$ we partition it in petals. For this we define a *Petals tree*. Recall that a tree is a directed acyclic graph with exactly one path from the root to each vertex.

DEFINITION (Petals tree). A Petals tree for $\widehat{C}_{\widehat{A}}(n, n)$ consists of a root $[r]$ that branches out in a subgraph of $\widehat{C}_{\widehat{A}}(n, n)$ including all its vertices. It has height n, the vertices at distance d to the root have exactly d occurrences of the new symbol σ, for $d = 0, \ldots, n$. The root $[r]$ is a necklace that corresponds to an Eulerian cycle in $G_A(n - 1, n)$.

There are many Petals trees for $\widehat{C}_{\widehat{A}}(n, n)$, any one is good for our purpose. A Petals tree can be obtained by any algorithm that finds a spanning tree of a graph, as Kruskal's greedy algorithm for the minimal spanning tree, or it can be constructed using the classical Breath First search on $\widehat{C}_{\widehat{A}}(n, n)$.

We now focus on how to insert the petals in the given Eulerian cycle in $G_A(n - 1, n)$ but satisfying the small gap condition. Since each vertex u of $G_A(n-1, n)$ occurs exactly $|A|$ times in the given Eulerian cycle, we have $|A|$ many possibilities to place the petal for u. To determine where to place it, we divide the given Eulerian cycle in as many consecutive *sections* as the number of vertices in the graph $G_A(n-1, n)$. We say that an Eulerian cycle is pointed when there is a designated first edge.

DEFINITION (Section of a cycle). For a pointed Eulerian cycle in $G_A(n-1,n)$ given by the sequence of edges $e_1, \ldots, e_{n|A|^n}$ and an integer j such that $1 \le j \le n|A|^{n-1}$, the jth section of the cycle is the sequence of the $|A|$ vertices that are heads of $e_{j|A|}, \ldots, e_{j|A|+|A|-1}$.

We would like to choose one vertex from each section to place a petal. The difficulty is that each vertex occurs $|A|$ times in the Eulerian cycle but not necessarily at $|A|$ different sections. We pose a *matching* problem.

DEFINITION (Distribution graph). Given pointed Eulerian cycle in $G_A(n-1,n)$ the Distribution graph $D_A(n-1,n)$ is a $|A|$-regular bipartite graph, one part consists of the vertices in $G_A(n-1,n)$, the other part consists of the sections of the Eulerian cycle. There is an edge from a vertex u in $G_A(n-1,n)$ to a section j if u belongs to the section j.

A matching in a Distribution graph is a set of edges such that no two edges share a common vertex. A vertex is matched if it is an endpoint of one of the edges in the matching. A *perfect matching* is a matching that matches all vertices in the graph.

PROPOSITION 8. *For every Distribution graph $D_A(n-1,n)$ there is a perfect matching.*

P r o o f. Consider a finite bipartite graph consisting of two disjoint sets of vertices X and Y with edges that connect a vertex in X to a vertex in Y. For a subset W of X, let $N(W)$ be the set of all vertices in Y adjacent to some element in W. Hall's marriage theorem [13] states that there is a matching that entirely covers X if and only if for every subset W in X, $|W| \le |N(W)|$.

Consider a Distribution graph $D_A(n-1,n)$ and call X to the set of vertices $G_A(n-1,n)$ and Y to the set of sections. For any $W \subseteq X$ such that $|W| = r$, the sum of the out-degree of these r vertices is $r|A|$. Since the in-degree of each vertex in Y is $|A|$, we have that $|N(W)| \ge r$. Then, there is a matching that entirely covers X. Furthermore, since the number of vertices is equal to the number of sections, $|X| = |Y|$, the matching is perfect. $\qquad\square$

To obtain a perfect matching in a Distribution graph we can use any method to compute the maximum flow in a network. We define the flow network by adding two vertices to the Distribution graph, the source and the sink. Add an edge from the source to each vertex in X and add an edge from each vertex in Y to the sink. Assign capacity 1 to each of the edges of the flow network. The maximum flow of the network is $|X|$. This flow has the edges of a perfect match.

We have the needed tools for the awaiting proof.

Proof of Lemma 1 (M a i n l e m m a). Assume $[v]$ is a (n,n)-perfect necklace over the alphabet A. We construct a (n,n)-perfect necklace $[\widehat{v}]$ over the alphabet $\widehat{A}$. By Proposition 2, we need to construct an Eulerian cycle in $G_{\widehat{A}}(n-1,n)$.

Consider a pointed Eulerian cycle in $G_A(n-1,n)$ with starting edge determined by v and consider its $n|A|^{n-1}$ sections. By Proposition 8 we choose one vertex in each section according to a perfect matching. Fix any Petals tree for $C_{\widehat{A}}(n,n)$ with root $[v]$. The construction traverses the graph $G_{\widehat{A}}(n-1,n)$ until it obtains an Eulerian cycle in this graph. The construction inserts one petal in each section of the Eulerian cycle in $G_A(n-1,n)$. All the sections are considered, one after the other, until all sections have been considered.

The construction starts at section 0, and at each step of the construction there is a current section. The traversal of of $G_{\widehat{A}}(n-1,n)$ starts at the starting vertex of the pointed Eulerian cycle in $G_A(n-1,n)$. Each time an edge is traversed, the current vertex becomes the edge's endpoint. Let (u,m) be the current vertex.

Case (u,m) is a vertex in $G_A(n-1,n)$. If (u,m) is the chosen vertex in the current section and the petal for (u,m), which starts with $[u\sigma,m]$ in $\widehat{C}_{\widehat{A}}(n,n)$, has not been inserted yet, then insert it now: traverse the edge that adds the symbol σ and continue traversing the edges in $G_{\widehat{A}}(n-1,n)$ corresponding to the petal for (u,m). If the petal for (u,m) has already been inserted or (u,m) is not a chosen vertex, then continue with the traversal of edges corresponding to the current section. If the current section is exhausted, the next section becomes the current section.

Case (u,m) is not a vertex in $G_A(n-1,n)$. If $[u\sigma,m]$ is a child of the current vertex in the Petals tree and it has not been inserted yet, then traverse the edge that adds the symbol σ and continue traversing the edges in $G_{\widehat{A}}(n-1,n)$ corresponding to the petal for (u,m). Otherwise continue with the traversal of the edges corresponding to the petal that (u,m) was already part of.

Finally, we prove the minimal gap condition. Obviously, each section of the Eulerian cycle in $G_A(n-1,n)$ has no occurrence of the symbol σ. A petal for a vertex (u,m) in $G_A(n-1,n)$ necessarily starts with the edge that adds the symbol σ right after u, and this petal corresponds to a path in $\widehat{C}_{\widehat{A}}(n,n)$. Since each section has $|A|$ edges, if we place one petal in each section, then two consecutive petals are at most $2|A|-1$ edges away. Pick a section and let (u,m) be the chosen vertex and let (u',m') be the chosen vertex in the next section. In case the petal for (u,m) corresponds just the single vertex $[u\sigma,m]$ in $\widehat{C}_{\widehat{A}}(n,n)$, then it is a cycle in $G_{\widehat{A}}(n-1,n)$ consisting of exactly n edges. So, in between the occurrence of σ in the petal for (u,m) and the first occurrence of σ in the

petal for (u', m') there are at most

$$n - 1 + 2|A| - 1 = n + 2|A| - 2 \quad \text{other symbols.}$$

In case, the petal for (u, m) consists of more than one vertex in $\widehat{C}_{\widehat{A}}(n, n)$, then before completing the traversal of the n edges corresponding to $[u\sigma, m]$, the construction:

(1) first branches out to another vertex in $\widehat{C}_{\widehat{A}}(n, n)$ and

(2) then traverses the corresponding edges in $G_{\widehat{A}}(n - 1, n)$ that relate vertices having at least one occurrence of the symbol σ,

(3) returns back to (u, m), necessarily from a vertex $(\sigma w, (m - 1) \bmod n)$, where w is the prefix of u of length $n - 2$.

So, in between the last occurrence of σ in the petal for (u, m) and the first occurrence of σ in the petal for (u', m') there are at most

$$n - 1 + 2|A| - 1 = n + 2|A| - 2 \quad \text{other symbols.}$$

It remains to argue what happens inside a petal. The vertices in $\widehat{C}_{\widehat{A}}(n, n)$ correspond to edges in the $G_{\widehat{A}}(n-1, n)$ that relate vertices with at least one occurrence of the symbol σ. This ensures that in between any two successive occurrences of σ inside a petal there are at most

$$n - 1 \quad \text{symbols.}$$

We conclude that in the traversal of $G_{\widehat{A}}(n - 1, n)$ in between any occurrence of σ and the next there are at most

$$n + 2|A| - 2 \quad \text{other symbols.} \qquad \square$$

EXAMPLE. Fix

$$A = \{0, 1\} \quad \text{and} \quad \widehat{A} = \{0, 1, \sigma\}.$$

Let $[v]$ be the n-ordered necklace for $n = 2$,

$$[v] = [00 \ 01 \ 10 \ 11].$$

Fix an Eulerian cycle for $[v]$ in $G_A(1, 2)$.
Since $G_A(1, 2)$ has 4 vertices, divide it in 4 sections:

- Section 0 contains the vertices $(0, 0)$ and $(0, 1)$.
- Section 1 contains the vertices $(0, 0)$ and $(1, 1)$.
- Section 2 contains the vertices $(1, 0)$ and $(0, 1)$.
- Section 3 contains the vertices $(1, 0)$ and $(1, 1)$.

The following choice gives a perfect match:

- Section 0 : $(0, 0)$,
- Section 1 : $(1, 1)$,
- Section 2 : $(0, 1)$,
- Section 4 : $(1, 0)$.

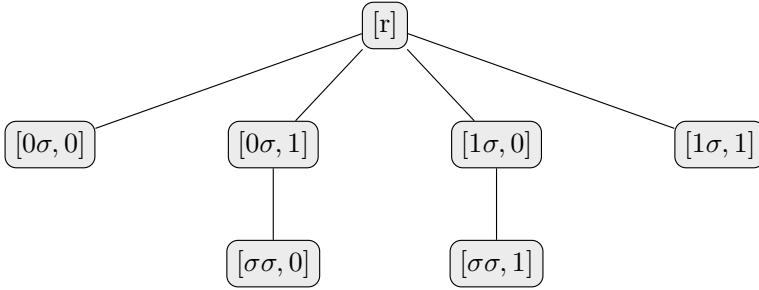Consider the following Petals tree with root $[r] = [00011011]$.

FIGURE 1. The Petals tree.

This Petals tree (see Fig. 1) has 4 branches, each one is a petal for a vertex in $G_A(1, 2)$:

(1) The first branch is a petal for $(0, 0)$, It results in the sequence $\sigma 0$ to be inserted (right after the 0 that appears at an even position).

(2) The second branch is a petal for $(0, 1)$. It is the join of two vertices in the tree, which results in the sequence $\sigma\sigma 0$ to be inserted (right after the 0 that appears at an odd position).

(3) The third branch is a petal for $(1, 0)$. It results in the sequence $\sigma\sigma 1$ to be inserted (right after the 1 that appears at an even position).

(4) The fourth branch is a petal for $(1, 1)$. It results in the sequence $\sigma 1$ to be inserted (right after the 1 that appears at an odd position).

The construction follows the Eulerian cycle for $[v]$ in $G_A(1, 2)$ and, in each section it inserts the petal for the chosen vertex for a perfect match, immediately after it. Thus:

- in section 0 it inserts the edges for $[0\sigma, 0]$ at $(0, 0)$,
- in section 1 it inserts the edges for $[1\sigma, 1]$ at $(1, 1)$,
- in section 2 it inserts the edges for $[0\sigma, 1]$ and $[\sigma\sigma, 0]$ at $(0, 1)$ and
- in section 3 it inserts the edges for $[1\sigma, 0]$ and $[\sigma\sigma, 0]$ at $(1, 0)$.

The result is the $(1, 2)$-perfect necklace $[\widehat{v}]$ over the alphabet $\widehat{A}$. The inserted symbols are in boldface:

$$[\widehat{v}] = [0\sigma\ \mathbf{0}0\ 01\ \sigma\mathbf{1}\ \ 10\ \sigma\sigma\ \mathbf{0}1\ \sigma\sigma\ \mathbf{1}1].$$

It satisfies the small gap condition because in between any occurrence of σ and the next there are at most 4 other symbols, which is less that the allowed because $n = 2$, $|A| = 2$ and $n + 2|A| - 2 = 5$ symbols.

3.2. Proof of Theorem 1

Suppose $v \in A^\omega$ is the concatenation of (n, n)-perfect necklaces over the alphabet A, for $n = 1, 2, \ldots$ Apply the Lemma 1 (Main lemma) to each of these (n, n)-perfect necklaces over the alphabet A and obtain (n, n)-perfect necklaces over the alphabet $\widehat{A}$. By Proposition 6, their concatenation is normal to the alphabet $\widehat{A}$.

Fix a positive integer N. Recall that the length of a (n, n)-perfect necklace over the alphabet $\widehat{A}$ is $n|\widehat{A}|^n$. Let m be the smallest integer such that such that

$$N \le \sum_{i=1}^{m} i|\widehat{A}|^i.$$

Therefore, $|\widehat{A}|^m < N$, hence, $m \le \log_{|\widehat{A}|} N$.

Consider the possibilities for the occurrences of σ in $\widehat{v}$ just before and just after position N. We need to analyze two cases.

Case they are both inside the same (n, n)-perfect necklace. Lemma 1 (Main lemma) proved that the number of symbols in between is less than $2|A|+n$, henceforth less than $2|A| + m$.

Case they are not in same perfect necklace. Notice that in the proof of Lemma 1 (Main lemma) the Eulerian cycle over alphabet A is divided in sections of size $|A|$, independently of the value of n. Assume that N is in the (m, m)-perfect necklace over $\widehat{A}$. First suppose that the occurrence before position N is in the $(m-1, m-1)$-perfect necklace. The construction in Lemma 1 ensures that σ occurs in the last $|A| + m$ symbols of this necklace and the next occurrence of σ is in the first $|A| + 1$ symbols of the (m, m)-perfect necklace. Thus, in between these two occurrences of σ there are at most $2|A|+m-1$ symbols. Now suppose that the occurrence after position N is in the $(m + 1, m + 1)$-perfect necklace. Then, there is an occurrence of σ in the last $|A| + m + 1$ symbols of the (m, m)--perfect necklace and the next occurrence of σ is in the first $|A|+1$ symbols of the $(m + 1, m + 1)$-perfect necklace. Therefore, in between the two occurrences of σ are at most $2|A| + m$ symbols.

Since $m \leq \log_{\widehat{A}} N$, it follows that for every N the number of symbols in between these occurrences of σ before an after position N is at most

$$2|A| + \log_{|\widehat{A}|} N.$$

This concludes the proof of Theorem 1. $\hfill\square$

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