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M. LEVIN'S CONSTRUCTION OF ABSOLUTELY NORMAL NUMBERS WITH VERY LOW DISCREPANCY

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ABSTRACT. Among the currently known constructions of absolutely normal numbers, the one given by Mordechay Levin in 1979 achieves the lowest discrepancy bound. In this work we analyze this construction in terms of computability and computational complexity. We show that, under basic assumptions, it yields a computable real number. The construction does not give the digits of the fractional expansion explicitly, but it gives a sequence of increasing approximations whose limit is the announced absolutely normal number. The n-th approximation has an error less than 2^{-2^n}. To obtain the n-th approximation the construction requires, in the worst case, a number of mathematical operations that is doubly exponential in n. We consider variants on the construction that reduce the computational complexity at the expense of an increment in discrepancy.

1. Introduction

Normal numbers were introduced by Émile Borel in 1909 [9]. A real number α is normal to an integer base λ greater than or equal to 2 if its fractional expansion in base λ given by

$$\alpha - \lfloor \alpha \rfloor = \sum_{k\geq 1} \frac{d_k}{\lambda^k},$$

where each d_k is in $\{0, 1, \dots, \lambda-1\}$, is such that, for each positive integer L, each fixed block of digits of length L appears in $(d_k)_{k\geq 1}$ with asymptotic frequency λ^{-L}. Borel calls a number absolutely normal if it is normal to every integer base greater than or equal to 2. Let $(\xi_k)_{k\geq 0}$ be an arbitrary sequence of real numbers in the unit interval:

$$D(N, (\xi_k)_{k\geq 0}) = \sup_{0\leq u<v\leq 1} \left| \frac{\#\{k : 0 \leq k < N \text{ and } u \leq \xi_k < v\}}{N} - (v-u) \right|$$

is the discrepancy of $(\xi_k)_{k=0}^{N-1}$. The sequence $(\xi_k)_{k\geq 0}$ is uniformly distributed in the unit interval if $D(N, (\xi_k)_{k\geq 0})$ goes to 0 when N goes to infinity. By a theorem of

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D. Wall [10, Theorem 4.14], a real number α is normal to base λ if, and only if, the sequence $\{\alpha\lambda^k\}_{k\geq 0}$, where $\{\xi\} = \xi - \lfloor\xi\rfloor$ is the fractional part of ξ, is uniformly distributed in the unit interval.

We use the customary notation for asymptotic growth of functions and we say $f(n)$ is in $O(g(n))$ if $\exists k > 0\ \exists n_0\ \forall n > n_0$, $|f(n)| \leq k|g(n)|$.

Borel [9] proved that almost every real number (in the sense of Lebesgue measure) is normal to every integer base and Gaal and Gál [14] showed that, indeed, for almost every real number α and for every integer base λ the discrepancy $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ is in $O\left(\sqrt{\frac{\log\log N}{N}}\right)$. For a thorough presentation of normal numbers and the theory of uniform distribution see the books [10, 13, 16].

In 1979 Mordechay Levin [18] considered the notion of normality for real numbers with respect to bases that are real numbers greater than 1: a real number α is normal with respect to a real base λ if the condition in Wall's theorem holds, that is, if the sequence $\{\alpha\lambda^k\}_{k\geq 0}$ is uniformly distributed in the unit interval. Levin gave an explicit construction of a number that is normal to countably many real bases, with controlled discrepancy of normality. More precisely, given a sequence $(\lambda_j)_{j\geq 1}$ of real numbers greater than 1, a monotone increasing sequence $(t_j)_{j\geq 1}$ of positive integers and a non-negative real number a, Levin constructs a real number α greater than a that is normal to each of the bases λ_j, for $j = 1, 2, \ldots$ such that $D(N, \{\alpha\lambda_j^k\}_{k\geq 0})$ is in $O\left(\frac{(\log N)^2}{\sqrt{N}}\omega(N)\right)$, where $\omega(N)$ is a non-decreasing unbounded function determined from $(\lambda_j)_{j\geq 1}$ and $(t_j)_{j\geq 1}$. For a convenient choice of $\omega(N)$, $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ ends up being in $O\left(\frac{(\log N)^3}{\sqrt{N}}\right)$. With $\lambda_j = j + 1$ for $j = 1, 2, \ldots$. Levin obtains a number α that is absolutely normal in Borel's sense.

A particular interest of this construction by Levin is that, among the currently known methods to construct absolutely normal numbers, it achieves the lowest discrepancy bound. In the present note we give a plain presentation of Levin's work [18] and analyze it in terms of computability and computational complexity. Section 2 constructs a real number α that is absolutely normal with respect to each base in a given countable set of real bases greater than 1. Section 3 shows that Levin's construction does not give the digits of the fractional expansion of the number α explicitly, but it gives a sequence of increasing approximations with limit α. The n-th approximation has an error less than $2^{2^{-n}}$. We prove that Levin's construction cannot be modified to produce directly the digits of α, one after the other. We also conclude that any change in the construction implying a faster computation would necessarily yield a larger discrepancy associated to the absolutely normal number. Section 4 proves that, for basic assumptions on the starting elements, Levin's construction yields an algorithm to compute the number α. Finally Section 5 analyzes the computational complexity of the algorithm in terms of the number of mathematical operations needed in the computation. To obtain the n-th approximation to the number α the construction requires, in the worst case, a number of mathematical operations that are doubly exponential in n.

About known constructions of absolutely normal numbers. With the exception of this work by Levin, known constructions of absolutely normal numbers considered explicitly the computational complexity but they did not give a closed formula for the discrepancy. The recent work of Adrian-Maria Scheerer [21] provides these missing calculations. Thus, regarding discrepancy and computational complexity, known constructions of computable absolutely normal numbers can be classified as follows:

- Constructions that run in doubly exponential time, which means that to produce the N-th digit of the expansion of the constructed number α in a given base they perform a number of operations that are doubly exponential in N. One example is Alan Turing's algorithm [4, 24] for which $D(N, \{\alpha\lambda^{c2^{2k+1}}\}_{k\geq 0})$ is in $O\left(\frac{1}{\sqrt[16]{N}}\right)$, proved in [21, section A.4]. Another is the computable reformulation of Sierpiński's construction [3] for which $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ is in $O\left(\frac{1}{\sqrt[6]{N}}\right)$, proved in [21, section A.1].
- Constructions that run in exponential time, as Wolfgang Schmidt's algorithm [23] for which $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ is in $O\left(\frac{\log\log N}{\log N}\right)$. Scheerer [21] gives this discrepancy bound and he presents a modification of Schmidt's algorithm such that, for any fixed positive A, it computes a number that depends on A with associated discrepancy $O\left((\log N)^{-A}\right)$. The variants of Schmidt's algorithm given by Becher, Bugeaud and Slaman [2, 7] also require exponential time. These algorithms produce numbers that are normal to all the bases in a given arbitrary set, while they are not (simply) normal to any of the multiplicatively independent bases in the complement. Furthermore, the algorithm for computing an absolutely normal Liouville number α given by Becher, Heiber and Slaman [5] has at least exponential complexity and we have not estimated the discrepancy of the sequence $\{\alpha\lambda^k\}_{k=0}^{N-1}$, for positive N.
- Constructions that run in polynomial time, as the algorithm given by Becher, Heiber and Slaman [6] which computes an absolutely normal number α with just above quadratic complexity (to produce the N-th digit of α it performs a number of operations just above quadratic in N). Speed of computation is obtained by sacrificing discrepancy. The algorithm deals explicitly with the discrepancy at the intermediate steps of the construction but we have not estimated the discrepancy of the sequence $\{\alpha\lambda^k\}_{k\geq 0}$.

About constructions ensuring normality to just one base. There are constructions of numbers ensuring normality to just one base which achieve much lower discrepancy bounds than those for absolute normality. The one with smallest discrepancy was given also by Levin [19] using van der Corput type sequences. Levin constructs a number α normal to an integer base λ, such that the discrepancy $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ is in $O\left(\frac{(\log N)^2}{N}\right)$. This discrepancy bound is surprisingly small, considering that Schmidt proved (see [10]) that for any sequence $(\xi_k)_{k\geq 0}$ of reals in the unit interval, $\limsup_{N\to\infty} D(N, (\xi_k)_{k\geq 0}) \geq \frac{1}{25}\frac{\log N}{N}$. The computational complexity of this construction by Levin has not been studied yet. Recently, Manfred Madritsch and Robert Tichy [20] found conditions for van der Corput sets and suggested using them for constructions ensuring normality not just to a single base, but to all integer bases. This line of investigation seems worth studying.

The construction ensuring normality to one base that has essentially the smallest computational complexity coincides with the historically first construction of a number that is normal to base 10, due to David Champernowne in 1933 [11]. Champernowne's constant is the number in the unit interval whose decimal expansion is the concatenation of the positive integers in their natural order:

$$0.1234567891011121314151617\dots.$$

It is computable with logarithmic complexity, which means that the N-th digit in the expansion can be obtained independently of all the previous digits by performing $O(\log N)$ elementary operations. It is also possible to compute the first N digits of Champernowne's constant in $O(N)$ operations. The discrepancy $D(N, \{\alpha\lambda^k\}_{k\geq 0})$ is in $O\left(\frac{1}{\log N}\right)$ and it has been proved (see [11, 19, 22]) that there is a positive K such that for every N, $D(N, \{\alpha\lambda^k\}_{k\geq 0}) \geq \frac{K}{\log N}$.

2. Levin's construction

In this section we give a comprehensible presentation of Levin's construction [18]. Hereafter we use the star-discrepancy, which is similar to discrepancy but it is defined in terms of intervals $[0, \gamma)$ for $0 < \gamma \leq 1$, instead of intervals $[u, v)$ for $0 \leq u < v \leq 1$. For any positive integer N and for any sequence $(\xi_k)_{k\geq 0}$ of real numbers in the unit interval,

$$D^*(N, (\xi_k)_{k\geq 0}) = \sup_{\gamma\in(0,1]} \left| \frac{\#\{k : 0 \leq k < N \text{ and } \xi_k < \gamma\}}{N} - \gamma \right|.$$

The two notions differ at most by a constant factor (see [16]) because

$$D^* \leq D \leq 2D^*.$$

Definition. Let λ be a real number greater than 1 and let $\Lambda = (\lambda_j)_{j=1}^{\infty}$ be a sequence of real numbers, each greater than 1. A number α is normal to base λ if the sequence $\{\alpha\lambda^k\}_{k\geq 0}$ is uniformly distributed in the unit interval, and Λ-absolutely normal, if α is normal to base λ_j for each positive j.

Theorem 1 (Levin [18])**.** *Let $\Lambda = (\lambda_j)_{j\geq 1}$ be a sequence of real numbers greater than 1, let $(t_j)_{j\geq 1}$ be a sequence of integers monotonically increasing at any speed and let a be a non-negative real number. There is a real number α constructed from a and the sequences $(\lambda_j)_{j\geq 1}$ and $(t_j)_{j\geq 1}$ which is Λ-absolutely normal and such that for any positive integer N,*

$$D^*(N, \{\alpha\lambda_j^k\}_{k\geq 0}) \text{ is in } O\left(\frac{(\log N)^2\omega(N)}{\sqrt{N}}\right),$$

where $\omega(N) = 1$ if $N \in [1, \ell_2)$, and $\omega(N) = k$ if $N \in [\ell_k, \ell_{k+1})$, with $\ell_k = \max(t_k, \max_{1\leq v\leq k} 2\lceil |\log_2 \log_2 \lambda_v|\rceil + 5)$ and the constant in the order symbol depends on λ_j.

The real number α proposed by Levin is defined as

$$\alpha = a + \sum_{r=\ell_1}^{\infty} \frac{a_r}{2^{n_r} q_r},$$

where

$n_r = 2^r - 2$,
$q_r = 2^{2^r+r+1}$, and
a_r is an integer between 0 and q_r that satisfies the property stated in Lemma 4, which is used in the proof of Theorem 1.

Fix $(\lambda_j)_{j\geq 1}$ an arbitrary sequence of real numbers greater than 1, fix $(t_j)_{j\geq 1}$ a sequence of integers monotonically increasing at any speed and fix a non-negative real a. Along this note we refer freely to the values ℓ_r, n_r, q_r, a_r and $\omega(r)$ for any positive r as well as to the real α.

We need some further notation. For each pair of positive integers r, j we let

$n_{r,j} = \lfloor n_r \log_{\lambda_j} 2 \rfloor$,
$\tau_{r,j} = n_{r+1,j} - n_{r,j}$, and
$A_{r,j} = \lfloor \sqrt{\tau_{r,j}} \rfloor$.

Before the proof of Theorem 1 we give Lemmas 2, 3, 4 and 5.

Lemma 2. *For every positive j and for every $r \geq \ell_j - 1$,*

$2^{r-1} \log_{\lambda_j} 2 \leq \tau_{r,j} \leq 2^{r+1} \log_{\lambda_j} 2$ *and*
$\tau_{r,j} \geq \max(7, \tau_{r+1,j}/4)$.

Proof. From the definitions we know that $\tau_{r,j} = 2^r \log_{\lambda_j} 2 + \theta_{r,j}$, where $|\theta_{r,j}| \leq 1$, while for $r \geq \ell_j - 1$ we have $8 = 2^{\log_2 \log_2 \lambda_j + 3} \log_{\lambda_j} 2 \leq 2^r \log_{\lambda_j} 2$. The wanted inequalities follow. □

Fix $\alpha_{\ell_1} = a$ and for each positive integer m, let a_m in $[0, q_m)$. For every $r \geq \ell_1$,

$$\alpha_{r+1} = \alpha_{\ell_1} + \sum_{m=\ell_1}^{r} \frac{a_m}{2^{n_m} q_m}.$$

We write $e(x)$ to denote $e^{2\pi i x}$. For integers c, m_1, m_2, r with $r \geq \ell_j$ we define

$$S_{r,j}(m_1, m_2, c) = \sum_{k=0}^{\tau_{r,j}-1} e\left(m_1 \left(\alpha_r + \frac{c}{2^{n_r} q_r} \right) \lambda_j^{n_{r,j}+k} + \frac{m_2 k}{\tau_{r,j}} \right),$$

$$D_{r,j}(c) = \sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{|S_{r,j}(m_1, m_2, c)|}{\overline{m_1}\;\overline{m_2}},$$

where $\overline{m} = \max(1, |m|)$ and $\sum'$ denotes that the term with $m_1 = m_2 = 0$ is absent from the sum.

Remark. In [18] the definition of $S_{r,j}(m_1, m_2, c)$ appears with $\sum'$ and the definition of $D_{r,j}(c)$ appears with $\sum$. We corrected these because $\sum'$ excludes the term $m_1 = m_2 = 0$, which only makes sense in the definition of $D_{r,j}(c)$.

Lemma 3 (Lemma 1 in [18]). *Let integers j, r, m_1, m_2 such that $r \geq \ell_j$ and $0 < \max(|m_1|, |m_2|) \leq A_{r,j}$. Then,*

$$\left(\frac{1}{q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1, m_2, c)|^2 \right)^{1/2} < 2 \left(\frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}}.$$

Construction: M.Levin's construction of absolutely normal numbers

Input : **a sequence** $(\lambda_j)_{j\geq 1}$ **of reals greater than** 1**; an increasing sequence** $(t_j)_{j\geq 1}$ **of integers; a non-negative real** a**.**

Output: **a sequence of rationals** $(\alpha_r)_{r\geq 1}$ **such that** $\lim_{r\to\infty} \alpha_r = \alpha$ **and for each** λ_j**, the discrepancy of** $\{\alpha\lambda_j^k\}_{k=0}^N$ **is in** $O\left(\frac{(\log N)^2\omega(N)}{\sqrt{N}}\right)$**.**

Define the function $\ell_k = \max(t_k, \max_{1\leq j\leq k} 2\lceil|\log_2\log_2\lambda_j|\rceil + 5)$

$r = \ell_1$

$\alpha_r = a$

repeat *forever*

$n_r = 2^r - 2$

$q_r = 2^{2^r+r+1}$

if r **in** $[1, \ell_2)$ **then** $\omega(r) = 1$

else $\omega(r) =$ **the unique** k **such that** r **in** $[\ell_k, \ell_{k+1})$

for $j = 1$ **to** $\omega(r)$ **do**

$\tau_{r,j} = n_{r+1,j} - n_{r,j}$

$A_{r,j} = \lfloor\sqrt{\tau_{r,j}}\rfloor$

end

find the least integer a_r **in** $[0, q_r)$ **such that for each** j **in** $[1, \omega(r)]$

$$D_{r,j}(a_r) < 2\left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2}\sqrt{\tau_{r,j}}\,(3 + \ln\tau_{r,j})^2$$

where

$$D_{r,j}(c) = \sideset{}{'}\sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}} \frac{|S_{r,j}(m_1, m_2, c)|}{\overline{m_1}\ \overline{m_2}},$$

$$S_{r,j}(m_1, m_2, c) = \sum_{k=0}^{\tau_{r,j}-1} e\left(m_1\left(\alpha_r + \frac{c}{2^{n_r}q_r}\right)\lambda_j^{n_{r,j}+k} + \frac{m_2 k}{\tau_{r,j}}\right),$$

$\sum'$ **denotes the sum without the term with** $m_1 = m_2 = 0$**,**

$\overline{m} = \max(1, |m|)$**.**

$\alpha_{r+1} = \alpha_r + \frac{a_r}{2^{n_r}q_r}$

print α_{r+1}

$r = r + 1$

end

Proof. Let $T_{r,j}(m_1,m_2) = \left(\frac{1}{q_r}\sum_{c=0}^{q_r-1}|S_{r,j}(m_1,m_2,c)|^2\right)^{1/2}$.

Remark. In [18] Levin uses $S_{r,j}(m_1,m_2)$. We changed it to the correct expression $S_{r,j}(m_1,m_2,c)$.

For a complex expression S, we write S^* for its complex conjugate. Then, the square of the absolute value of $S=\sum_{k=1}^{N} e(x_k)$ is

$$|S|^2 = S\cdot S^* = \sum_{k=1}^{N} e(x_k)\cdot\sum_{k=1}^{N} e(-x_k) = \sum_{k,j=1}^{N} e(x_k - x_j).$$

Then,

$$T^2_{r,j}(m_1,m_2) = \sum_{k,h=0}^{\tau_{r,j}-1}\frac{1}{q_r}\sum_{c=0}^{q_r-1} e\left(m_1\left(\alpha_r+\frac{c}{2^{n_r}q_r}\right)\left(\lambda_j^{n_{r,j}+k}-\lambda_j^{n_{r,j}+h}\right)+\frac{m_2(k-h)}{\tau_{r,j}}\right).$$

In accordance with the familiar inequality

$$\frac{1}{N}\left|\sum_{k=0}^{N-1} e(\theta k)\right| \le \min\left(1,\frac{1}{2N\langle\langle\theta\rangle\rangle}\right),$$

where $\langle\langle\theta\rangle\rangle$ is the distance of θ from the nearest integer, we have

$$\begin{aligned} T^2_{r,j}(m_1,m_2) &= \sum_{k,h=0}^{\tau_{r,j}-1}\frac{1}{q_r}\sum_{c=0}^{q_r-1} e\left(m_1\left(\alpha_r+\frac{c}{2^{n_r}q_r}\right)\left(\lambda_j^{n_{r,j}+k}-\lambda_j^{n_{r,j}+h}\right)+\frac{m_2(k-h)}{\tau_{r,j}}\right)\\ &< \sum_{k,h=0}^{\tau_{r,j}-1}\min\left(1,\frac{1}{2q_r\langle\langle m_1\frac{\lambda_j^{n_{r,j}+k}-\lambda_j^{n_{r,j}+h}}{2^{n_r}q_r}\rangle\rangle}\right).\end{aligned}$$

If m_1 equals 0, then m_2 does not belong to $0 \pmod{\tau_{r,j}}$, $\tau_{r,j}\ge 7$, $0<|m_2|\le A_{r,j}<\tau_{r,j}$, and $T_{r,j}(0,m_2)=0$. Let $|m_1|>0$. Let us show that the expression under the $\langle\langle\rangle\rangle$ sign above has absolute value less than $1/2$. Since $r\ge \ell_j$, by Lemma 2,

$$\begin{aligned} \lambda_j^{n_{r+1,j}} &\le \lambda_j^{n_{r+1}log_{\lambda_j}2} = 2^{n_{r+1}} = 2^{n_r}2^{2^r},\\ \log_{\lambda_j}2 &= 2^{-\log_2\log_2\lambda_j} < 2^{\ell_j-3} < 2^{r-3},\\ A_{r,j} &= \lfloor\sqrt{\tau}_{r,j}\rfloor < \sqrt{2^{r+1}\log_{\lambda_j}2} < 2^{r-1}.\end{aligned}$$

Hence,

$$|m_1(\lambda_j^{n_{r,j}+k}-\lambda_j^{n_{r,j}+h})| < 2A_{r,j}\lambda_j^{n_{r+1,j}} < 2^r2^{n_r}2^{2^r} = (1/2)2^{n_r}q_r,$$

and we can replace $\langle\langle\rangle\rangle$ by the absolute value sign:

$$T^2_{r,j}(m_1,m_2) \le \tau_{r,j} + 2\sum_{\tau_{r,j}>k>h\ge 0}\frac{2^{n_r}}{2|m_1|\lambda_j^{n_{r,j}}(\lambda_j^k-\lambda_j^h)}.$$

Using the definition of $n_{r,j}$,

$$\lambda_j^{n_{r,j}+1} \geq \lambda_j^{n_r \log_{\lambda_j} 2} = 2^{n_r}.$$

We obtain

$$\begin{aligned} T_{r,j}^2(m_1,m_2) &\leq \tau_{r,j} + \sum_{\tau_{r,j}>k>h\geq 0} \frac{1}{\lambda_j^h \lambda_j^{k-h-1}(1-\lambda_j^{h-k})} \\ &< \tau_{r,j} + \sum_{h,k=0}^{\infty} \frac{1}{\lambda_j^h \lambda_j^k (1-\lambda_j^{-1})} \\ &= \tau_{r,j} + \left(\frac{\lambda_j}{\lambda_j - 1}\right)^3 \\ &< 4\tau_{r,j}\left(\frac{\lambda_j}{\lambda_j-1}\right)^3. \end{aligned}$$

□

Lemma 4 ([18, Lemma 2]). *Let $r \geq \ell_1$. There exists an integer a_r in $[0, q_r)$ such that, given any positive integer j and with the condition $r \geq \ell_j$, we have*

$$D_{r,j}(a_r) < 2\left(\frac{\lambda_j}{\lambda_j-1}\right)^{3/2} \sqrt{\tau_{r,j}}\,(3+\ln \tau_{r,j})^2\,\omega(r).$$

Proof. Using the Cauchy-Bunyakovskii-Schwarz inequality we obtain

$$\begin{aligned} \frac{1}{q_r}\sum_{c=0}^{q_r-1} D_{r,j}(c) &= \sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{1}{\overline{m_1 m_2} q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1,m_2,c)| \\ &\leq \sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{1}{\overline{m_1 m_2}} \left(\frac{1}{q_r}\sum_{c=0}^{q_r-1} |S_{r,j}(m_1,m_2)|^2\right)^{1/2}. \end{aligned}$$

Since the conditions of Lemma 3 are satisfied, we have

$$\begin{aligned} \frac{1}{q_r}\sum_{c=0}^{q_r-1} D_{r,j}(c) &< 2\left(\frac{\lambda_j}{\lambda_j-1}\right)^{3/2} \sqrt{\tau_{r,j}}(3+2\ln A_{r,j})^2 \\ &\leq 2\left(\frac{\lambda_j}{\lambda_j-1}\right)^{3/2} \sqrt{\tau_{r,j}}(3+\ln \tau_{r,j})^2. \end{aligned}$$

Consequently, with $r \geq \ell_j$, the number of integers c in $[0, q_r)$ such that

$$D_{r,j}(c) \geq 2\omega(r)\left(\frac{\lambda_j}{\lambda_j-1}\right)^{3/2} \sqrt{\tau_{r,j}}(3+\ln \tau_{r,j})^2$$

is less than $q_r/\omega(r)$. By the definitions of $\omega(r)$ and ℓ_j, conditions $r \geq \ell_j$ and $\omega(r) \geq j$ are equivalent. In this case, the number of integers c in $[0, q_r)$, such that the above inequality holds for at least one positive integer j, with the condition $r \geq \ell_j$ (alternatively, $j \in [1, \omega(r)]$) is less than $\omega(r)\lfloor q_r/\omega(r)\rfloor = q_r$. Thus, there exists an integer $c = a_r$ in $[0, q_r)$, such that the inequality in the statement of this lemma holds for all positive integers j with the condition $r \geq \ell_j$. □

For the proof of Theorem 1 Levin uses multidimensional discrepancy and applies Erdös-Turán-Koksma's inequality [15].

Let s be a positive integer, let γ_v for $v = 1, \dots, s$ be real numbers in the unit interval, let $(\beta_{k,v})_{k\geq 0}$ for $v = 1, \dots, s$ be real number sequences, and let $C_v(N)$ be the number of solutions for $k = 0, 1, \dots, N-1$ of the system of inequalities

$$\begin{array}{rcl} \{\beta_{k,1}\} & < & \gamma_1 \\ \{\beta_{k,2}\} & < & \gamma_2 \\ & \vdots & \\ \{\beta_{k,s}\} & < & \gamma_s. \end{array}$$

The quantity

$$D^*(N, (\{\beta_{k,1}\}, \dots, \{\beta_{k,s}\})_{k\geq 0}) = \sup_{\gamma_1,\dots,\gamma_s \in (0,1]^s} \left| \frac{C_v(N)}{N} - \gamma_1 \cdot \dots \cdot \gamma_s \right|$$

is called the discrepancy of the sequences $\{\beta_{k,1}\}, \dots, \{\beta_{k,s}\}$, for $k = 0 \dots, N-1$.

Lemma 5 (Erdös-Turán-Koksma [15], [13, Theorem 1.21])**.** *Let s be a positive integer, let γ_v, for $v = 1, \dots, s$, be real numbers in the unit interval, let $(\beta_{k,v})_{k\geq 0}$ for $v = 1, \dots, s$ be a set of real number sequences. Let N be a positive integer. Then, for every integer n, the quantity $D^*(N, (\{\beta_{k,1}\}, \dots, \{\beta_{k,s}\})_{k\geq 0})$ is at most*

$$\left(\frac{3}{2}\right)^s \left(\frac{2}{n+1} + \frac{1}{N} \sum_{m_1 \dots m_s = -n}^{n}{}' \frac{\left| \sum_{k=0}^{N-1} e\left(\sum_{v=1}^{s} m_v \beta_{k,v}\right) \right|}{\overline{m_1} \dots \overline{m_s}} \right),$$

where $\sum{}'$ denotes that the term with $m_1 = m_2 = \dots = m_s = 0$ is absent from the sum, and $\overline{m} = \max(1, |m|)$.

Remark. Instead of the version of Erdös-Turán-Koksma inequality in Lemma 5, Levin uses in [18] the weaker version which states that, for every integer n, $D^*(N, (\{\beta_{k,1}\}, \dots, \{\beta_{k,s}\})_{k\geq 0})$ is at most

$$30^s \left(\frac{1}{n} + \frac{1}{N} \sum_{m_1 \dots m_s = -n}^{n}{}' \frac{\left| \sum_{k=0}^{N-1} e\left(\sum_{v=1}^{s} m_v \beta_{k,v}\right) \right|}{\overline{m_1} \dots \overline{m_s}} \right).$$

In the proof of Theorem 1 we use the stronger version but we obtain the same asymptotic expression for the discrepancy as that obtained by Levin.

Proof of Theorem 1. For any three real numbers ξ, λ, γ and non-negative integers M and N, we denote by $C_{\xi,\lambda,\gamma}(M, N)$ the number of solutions of the inequality

$$\{\xi\lambda^k\} < \gamma, \qquad \text{for } k = M, \dots, M+N-1.$$

We write $C_{\xi,\lambda,\gamma}(N)$, to denote $C_{\xi,\lambda,\gamma}(0, N)$. Fix any positive integer j and any positive real γ in the unit interval. Fix any positive integer N and define an integer h from the condition $n_{h,j} \leq N < n_{h+1,j}$. Then,

$$N = n_{h,j} + R_1, \text{ where } 0 \leq R_1 < \tau_{h,j}.$$

Observe that when N is large enough, $h \geq \ell_j$. Using the definition of $C_{\alpha,\lambda_j,\gamma}$,

$$C_{\alpha,\lambda_j,\gamma}(N) = C_{\alpha,\lambda_j,\gamma}(n_{\ell_j,j}) + \sum_{r=\ell_j}^{h} C_{\alpha,\lambda_j,\gamma}(n_{r,j}, \tau'_{r,j}),$$

where $\tau'_{r,j} = \tau_{r,j}$ for $r \in [\ell_j, h)$ and $\tau'_{h,j} = R_1$.

Remark. In [18] Levin uses n_{ℓ_j}. We changed it to the correct expression $n_{\ell_j,j}$.

Let us estimate $C_{\alpha,\lambda_j,\gamma}(n_{r,j},R)$ for $r \geq \ell_j$ and $0 \leq R \leq \tau_{r,j}$. The quantity $C_{\alpha,\lambda_j,\gamma}(n_{r,j},R)$ is equal to the number of solutions of the system of inequalities, for $k = 0,\dots,\tau_{r,j}-1$,

$$\left\{\frac{k}{\tau_{r,j}}\right\} < \frac{R}{\tau_{r,j}},$$
$$\{\alpha\lambda_j^{n_{r,j}+k}\} < \gamma.$$

We apply Lemma 5 with $s = 2$, $N = \tau_{r,j}$ and $n = A_{r,j}$ and obtain

$$\left|C_{\alpha,\lambda_j,\gamma}(n_{r,j},R) - \gamma\frac{R}{\tau_{r,j}}\tau_{r,j}\right| \leq \left(\frac{3}{2}\right)^2\left(\frac{2\tau_{r,j}}{A_{r,j}+1} + \sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{1}{\overline{m_1}\ \overline{m_2}}\left|\sum_{x=0}^{\tau_{r,j}-1} e\left(m_1\alpha\lambda_j^{n_{r,j}+x} + \frac{m_2 x}{\tau_{r,j}}\right)\right|\right).$$

Using the definition of α_r, we have that for any $r \geq \ell_1$,

$$\alpha = \alpha_r + \frac{a_r}{2^{n_r}q_r} + \frac{\theta_r}{2^{n_{r+1}}},$$

where $0 \leq \theta_r \leq 2$ because

$$\frac{\theta_r}{2^{n_{r+1}}} = \sum_{k=r+1}^{\infty}\frac{a_k}{2^{n_k}q_k} < \sum_{k=r+1}^{\infty}\frac{1}{2^{n_k}} = \frac{1}{2^{n_{r+1}}}\sum_{k=r+1}^{\infty}\frac{1}{2^{n_k-n_{r+1}}} \leq \frac{2}{2^{n_{r+1}}}.$$

By definition, $D_{r,j}(a_r) = \displaystyle\sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{|S_{r,j}(m_1,m_2,a_r)|}{\overline{m_1}\ \overline{m_2}}$, so

$$\left|C_{\alpha,\lambda_j,\gamma}(n_{r,j},R) - \gamma R\right| \leq \left(\frac{3}{2}\right)^2\left(\frac{2\tau_{r,j}}{A_{r,j}+1} + D_{r,j}(a_r) + \sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{1}{\overline{m_1}\ \overline{m_2}}|U(m_1,m_2,a_r)|\right)$$

where

$$|U(m_1,m_2,a_r)| = \left|S_{r,j}(m_1,m_2,a_r) - \sum_{k=0}^{\tau_{r,j}-1} e\left(m_1\alpha\lambda_j^{n_{r,j}+k} + \frac{m_2 k}{\tau_{r,j}}\right)\right|.$$

By the definition of $S_{r,j}(m_1,m_2,a_r)$, the condition $0 \leq \theta_r \leq 2$, and the fact that for every pair of reals ξ_1 and ξ_2,

$$|e(\xi_1) - e(\xi_2)| = 2|\sin(\pi(\xi_1-\xi_2))| \leq 2\pi|\xi_1-\xi_2|,$$

we find that

$$\begin{aligned}
|U(m_1, m_2, a_r)| &\le 2\pi \sum_{k=0}^{\tau_{r,j}-1} |m_1|\lambda_j^{n_{r,j}+k} \frac{\theta_r}{2^{n_{r+1}}} \\
&\le 4\pi|m_1|\lambda_j^{n_{r+1,j}} \frac{1}{(\lambda_j - 1)2^{n_{r+1}}} \\
&\le \frac{4\pi|m_1|}{\lambda_j - 1} \\
&\le \frac{4\pi A_{r,j}}{\lambda_j - 1} \\
&\le \frac{4\pi}{\lambda_j - 1}\sqrt{\tau_{r,j}}.
\end{aligned}$$

By the upper bound for $D_{r,j}(a_r)$ given in Lemma 4 for $r \ge \ell_j$, and the inequality $\sum_{m_1,m_2=-A_{r,j}}^{A_{r,j}}{}' \frac{1}{\overline{m_1 m_2}} \le (3 + \ln \tau_{r,j})^2$, we obtain that

$$\begin{aligned}
&|C_{\alpha,\lambda_j,\gamma}(n_{r,j}, R) - \gamma R| \\
&\quad \le \left(\frac{3}{2}\right)^2 \left(2\sqrt{\tau_{r,j}} + 2\left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{\tau_{r,j}}(3 + \ln \tau_{r,j})^2\omega(r)\right. \\
&\qquad\qquad \left. + \frac{4\pi}{\lambda_j - 1}\sqrt{\tau_{r,j}}(3 + \ln \tau_{r,j})^2\right) \\
&\quad \le \left(\frac{3}{2}\right)^2 15\left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{\tau_{r,j}}(3 + \ln \tau_{r,j})^2\omega(r).
\end{aligned}$$

Using $N = n_{h,j} + R_1$, where $0 \le R_1 < \tau_{h,j}$, and the equality for $h \ge \ell_j$,

$$C_{\alpha,\lambda_j,\gamma}(N) = C_{\alpha,\lambda_j,\gamma}(n_{\ell_j,j}) + \sum_{r=\ell_j}^{h} C_{\alpha,\lambda_j,\gamma}(n_{r,j}, \tau'_{r,j}),$$

we obtain that

$$\begin{aligned}
&|C_{\alpha,\lambda_j,\gamma}(N) - \gamma N| \\
&\quad \le |C_{\alpha,\lambda_j,\gamma}(n_{\ell_j,j}) - \gamma n_{\ell_j,j}| + \sum_{r=\ell_j}^{h} \left(\frac{3}{2}\right)^2 15\left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{\tau_{r,j}}(3 + \ln \tau_{r,j})^2\omega(r),
\end{aligned}$$

and, by Lemma 2,

$$\frac{1}{4}\tau_{h,j} \le \tau_{h-1,j} \le N.$$

Hence,

$$3 + \ln \tau_{r,j} \le 3 + \ln(4N) \le 5 + \ln N$$

and by definition of $\tau_{r,j}$,

$$\sum_{r=\ell_j}^{h} \sqrt{\tau_{r,j}} \le \sum_{r=\ell_j}^{h} \sqrt{2^{r+1}\log_{\lambda_j} 2} \le 3\sqrt{2^{h+2}\log_{\lambda_j} 2} \le 10\sqrt{\tau_{h,j}} \le 20\sqrt{N}.$$

Let us show that, for $h \geq \ell_j$, $\omega(N) \geq \omega(h)$. Since $\omega(r)$ is a non-decreasing sequence, it is sufficient to show that, for $h \geq \ell_j$, $N \geq h$. In fact, using the definitions of ℓ_j and $n_{h,j}$, and the equality $N = n_{h,j} + R_1$ we have for $h \geq 5$,

$$h \geq \ell_j \geq 5, \quad 2^{\frac{h+1}{2}} \geq h+1.$$

Thus,

$$\begin{aligned} N - h &\geq n_{h,j} - h \\ &\geq (2^h - 2)\log_{\lambda_j} 2 - h - 1 \\ &\geq (\log_{\lambda_j} 2)(2^{h-1} - (h+1)\log_2 \lambda_j) \\ &\geq (\log_{\lambda_j} 2)(2^{h-1} - (h+1)2^{\frac{\ell_j - 3}{2}}) \\ &\geq 2^{\frac{h-3}{2}}(\log_{\lambda_j} 2)(2^{\frac{h+1}{2}} - h - 1) \\ &\geq 0. \end{aligned}$$

Then, by the obvious inequality $|C_{\alpha,\lambda_j\gamma}(n_{\ell_j,j}) - \gamma n_{\ell_j,j}| \leq n_{\ell_j,j}$, we have

$$\begin{aligned} |C_{\alpha,\lambda_j,\gamma}(N) - \gamma N| &\leq n_{\ell_j,j} + \left(\frac{3}{2}\right)^2 \cdot 15 \cdot 20 \left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{N}(5 + \ln N)^2 \omega(N) \\ &\leq n_{\ell_j,j} + 675 \left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{N}(5 + \ln N)^2 \omega(N). \end{aligned}$$

The above inequality also holds for $h \leq \ell_j - 1$, since

$$|C_{\alpha,\lambda_j,\gamma}(N) - \gamma N| \leq N < n_{h+1,j} \leq n_{\ell_j,j}.$$

Recalling the definition of $n_{\ell_j,j}$ we finally obtain

$$|C_{\alpha,\lambda_j,\gamma}(N) - \gamma N| \leq 2^{\ell_j} \log_{\lambda_j} 2 + 675 \left(\frac{\lambda_j}{\lambda_j - 1}\right)^{3/2} \sqrt{N}(5 + \ln N)^2 \omega(N).$$

Hence, the discrepancy of the sequence $\{\alpha\lambda_j^k\}_{k\geq 0}$, for any given positive integer N,

$$D^*(N, \{\alpha\lambda_j^k\}_{k\geq 0}) = \sup_{\gamma\in(0,1]} \left| \frac{C_{\alpha,\lambda_j,\gamma}(N)}{N} - \gamma \right|$$

is in $O\left(\frac{(\log N)^2}{\sqrt{N}}\omega(N)\right)$. This completes the proof of Theorem 1. □

Corollary 6 ([18]). *Let $\lambda_j = j+1$, $t_j = 2^j$ for $j = 1, 2, \ldots$, so $\ell_j \leq 2^{j+1} + 1$ and $\omega(N) \leq 2(5 + \ln N)$. Then, the constructed number α is absolutely normal in Borel's sense, and for any integer $j \geq 2$, the discrepancy of $\{\alpha j^k\}$, for $k = 0, \ldots, N-1$ is*

$$D^*(N, \{\alpha j^k\}_{k\geq 0}) \leq \frac{2^{2^{j+1}+1}}{N} \log_j 2 + 1350 \frac{(5 + \ln N)^3}{\sqrt{N}},$$

which is in $O\left(\frac{(\log N)^3}{\sqrt{N}}\right)$.

Levin asserts that a similar method can be used for constructing a number α such that, given any integer j, the discrepancy of the sequence $\{\alpha\lambda_j^k\}_{k=0}^{N-1}$, is $O\left(\frac{(\log N)^{3/2}}{\sqrt{N}}\omega(N)\right)$, where the constant in the order symbol O depends on λ_j, and he gives as a reference [17, Section 2].

3. About Levin's construction and its possible variants

3.1. **Possible variants on the construction.** Here we consider other possible values for n_r and q_r to run Levin's construction. Observe that smaller values of q_r imply a faster computation at step r, because a_r is searched in a smaller range. However, smaller values of q_r imply slower growth of n_r, which in turn imply a larger discrepancy in the sequence $\{\alpha\lambda_j^k\}_{k\geq 0}$. Proposition 9 shows that it suffices that n_r grow quicker than r^h for $h > 1$ to ensure that Levin's construction yields an absolutely normal number. We first prove two lemmas.

Lemma 7. *If $\lambda_j \geq 2$ and the sequences $n_1, n_2, \ldots$ and $q_1, q_2, \ldots$ satisfy, for every positive r,*

$$2^{n_{r+1}-n_r+1+\frac{1}{2}\log(n_{r+1}-n_r+1)} \leq q_r,$$

then the statement of Lemma 3 holds.

Proof. In Lemma 3, every step of the proof is valid disregarding the values chosen for $n_1, n_2, \ldots$ and $q_1, q_2, \ldots$ except for the statement

$$|m_1|(\lambda_j^{n_{r,j}+k} - \lambda_j^{n_{r,j}+h}) \leq \frac{1}{2}2^{n_r}q_r.$$

We show that the condition given by this lemma is sufficient to make the above inequality true. Let us recall that $n_{r,j} = \lfloor n_r \log_{\lambda_j} 2\rfloor$, $\tau_{r,j} = n_{r+1,j} - n_{r,j}$, $0 \leq k, h < \tau_{r,j}$ and $|m_1| \leq A_{r,j} = \lfloor\sqrt{\tau_{r,j}}\rfloor$.

Then,

$$\begin{aligned}
q_r &\geq 2^{n_{r+1}-n_r+1+\frac{1}{2}\log_2(n_{r+1}-n_r+1)} = \sqrt{n_{r+1}-n_r+1}\; 2^{n_{r+1}-n_r+1} \\
&\geq \sqrt{(n_{r+1}\log_{\lambda_j} 2 - n_r \log_{\lambda_j} 2) + 1}\; 2^{n_{r+1}-n_r+1} \\
&\geq \sqrt{n_{r+1,j} - n_{r,j}}\; 2^{n_{r+1}-n_r+1} = \sqrt{\tau_{r,j}}\; 2^{n_{r+1}-n_r+1} \\
&\geq |m_1| 2^{n_{r+1}-n_r+1} = 2|m_1|2^{n_{r+1}}2^{-n_r} \\
&> 2|m_1|\lambda_j^{n_{r+1,j}}\lambda_j^{-(n_{r,j}+1)}2|m_1|\lambda_j^{n_{r+1,j}-n_{r,j}-1} = 2|m_1|\lambda_j^{\tau_{r,j}-1} \\
&> 2|m_1|(\lambda_j^{\tau_{r,j}-1} - 1) \\
&\geq 2|m_1|\frac{\lambda_j^{n_{r,j}}}{2^{n_r}}(\lambda_j^{\tau_{r,j}-1} - 1) \\
&\geq 2|m_1|\frac{\lambda_j^{n_{r,j}}}{2^{n_r}}(\lambda_j^{k} - \lambda_j^{h}) = \frac{2}{2^{n_r}}|m_1|(\lambda_j^{n_{r,j}+k} - \lambda_j^{n_{r,j}+h}).
\end{aligned}$$

□

In what follows we use customary asymptotic notation to describe the growth rate of the functions. We write

$f(n)$ is in $o(g(n))$	if $\forall k > 0\ \exists n_0\ \forall n > n_0$,	$\lvert f(n)\rvert \leq k\lvert g(n)\rvert$, and
$f(n)$ is in $\Theta(g(n))$	if $\exists k_1 > 0\ \exists k_2 > 0\ \exists n_0\ \forall n > n_0$,	$k_1 g(n) \leq f(n) \leq k_2 g(n)$.

Lemma 8. *Let j and N be positive integers and let k be such that $n_{k,j} \leq N < n_{k+1,j}$. If $\sum_{r=1}^{k}\sqrt{n_{r+1,j} - n_{r,j}}$ is in $o\left(\frac{N}{(\log N)^2\omega(N)}\right)$, then Levin's construction yields an absolutely normal number.*

Proof. See proof of Theorem 1 for the upper bound of $|C_{\alpha,\lambda_j,\gamma}(N) - \gamma N|$. □

The next proposition shows that if n_r dominates any linear function on r, and q_r is increasing in r according to a condition in the growth of n_r, then Levin's construction yields an absolutely normal number.

Proposition 9. *Let $(\lambda_j)_{j\geq 1}$ be a sequence of real numbers greater than 1 and let $(t_j)_{j\geq 1}$ be a sequence of reals such that the function $\omega(N)$ has sub-polynomial growth. If n_r grows quicker than r^h for $h > 1$ and q_r is such that*

$$n_{r+1} - n_r + 1 + \frac{1}{2}\log(n_{r+1} - n_r + 1) \leq \log q_r,$$

then Levin's construction yields an absolutely normal number. However, if n_r is linear in r, Levin's arguments do not prove that the discrepancy goes to 0.

Proof. Suppose n_r is polynomial on r. Then, there is some h such that n_r in $\Theta(r^h)$. By definition of $n_{r,j}$, we have $n_{r,j} = \lfloor n_r \log_{\lambda_j} 2\rfloor$ is in $\Theta(r^h)$. Hence, $n_{r+1,j} - n_{r,j}$ is in $\Theta(r^{h-1})$; therefore, $\sqrt{n_{r+1,j} - n_{r,j}}$ in $\Theta(r^{\frac{h-1}{2}})$. Furthermore, if N and k are such that $n_{k,j} \leq N < n_{k+1,j}$, then k is in $\Theta(\sqrt[h]{N})$. Thus,

$$\sum_{r=1}^{k} \sqrt{n_{r+1,j} - n_{r,j}} \text{ is in } \Theta\left(\left(\sqrt[h]{N}\right)^{\frac{h+1}{2}}\right) = \Theta\left(N^{\frac{h+1}{2h}}\right).$$

If n_r were a linear function on r, $\sum_{r=1}^{k} \sqrt{n_{r+1,j} - n_{r,j}}$ would be in $\Theta(N)$, hence $\sum_{r=1}^{k} \sqrt{n_{r+1,j} - n_{r,j}}$ would not be in the required class $o\left(\frac{N}{(\log N)^2\omega(N)}\right)$.

We conclude that, to obtain a normal number with Levin's construction, n_r can not be linear in r. Instead, n_r can be any polynomial on r with degree greater than 1 provided that $\omega(N)$ is chosen to have sub-polynomial growth. □

In Levin's construction smaller values of n_r imply a larger upper bound on discrepancy of the sequence $\{\alpha\lambda^k\}$. The following table shows the bound for the discrepancy of the sequence $\{\lambda_j^k\alpha\}_{k=0}^N$, obtained using Levin's proof for different choices of n_r. In each case the constant behind the O symbol depends on λ_j.

n_r	Discrepancy bound given by Levin's proof
r	$O(\log(N)^2\omega(N))$ —it does not go to 0 when N goes to ∞—
r^h	$O\left(\frac{\log(N)^2\omega(N)}{N^{\frac{h-1}{2h}}}\right)$
$2^r - 2$	$O\left(\frac{\log(N)^2\omega(N)}{\sqrt{N}}\right)$

In all of these cases, the upper bound for discrepancy contains $\omega(N)$, as in Levin's formulation and the constant hidden in the O symbol depends on the base λ_j. Although Levin stated that for any non-decreasing function $\omega(N)$ his construction produces an absolutely normal real number, the growth of $\omega(N)$ cannot be arbitrary. For example, when n_r is $2^r - 2$, $\omega(N) = \sqrt{N}$ does not give a discrepancy bound going to 0.

3.2. **Necessary conditions on the construction.** Levin's construction is not conceived as the concatenation of the binary expansions of the a_r for $r = 1, 2, \ldots$. This means that the expansion in base 2 of α_{r+1} is *not* obtained as a concatenation of the expansion of α_r with the base-2 representation of a_r. Recall the definition of α_{r+1}: α_{ℓ_1} is equal to a starting real number a (argument for the construction) and for every $r \geq \ell_1$,

$$\alpha_{r+1} = \alpha_{\ell_1} + \sum_{m=\ell_1}^{r} \frac{a_m}{2^{n_m} q_m},$$

where a_m is an integer in $[0, q_m)$ satisfying the conditions of Lemma 4,

$$n_m = 2^m - 2 \text{ and } q_m = 2^{2^m+m+1}.$$

Since $\log q_r = 2^r + r + 1 > n_{r+1} - n_r = 2^r$ we have

$$\alpha_{r+1} - \lfloor 2^{n_{r+1}} \alpha_{r+1} \rfloor 2^{-n_{r+1}} > 0.$$

In Levin's construction q_r and n_r are increasing in r and $q_r > 2^{n_{r+1}-n_r}$. This is necessary for the proof (it is not hard to check that without this condition the proof breaks) and it determines that Levin's construction of the number α *cannot* be achieved as the concatenation of the a_r, for $r = 1, 2, \ldots$.

Proposition 10. *If q_r and n_r are such that $\log q_r > n_{r+1} - n_r$, then Levin's construction of α is not achievable as the concatenation of the a_r, for $r = 1, 2, 3 \ldots$.*

Proof. To run the construction as a concatenation of the a_r, for $r = 1, 2, 3, \ldots$, we need that $\sum_{m=0}^{r-1} \log q_m \leq n_r$. But $\sum_{m=0}^{r-1} \log q_m > \sum_{m=0}^{r-1} n_{m+1} - n_m = n_r - n_0 = n_r$. $\square$

4. Levin's normal numbers are computable

The theory of computability defines a computable function from non-negative integers to non-negative integers as one which can be effectively calculated by some algorithm. The definition extends to functions from one countable set to another, by fixing enumerations of those sets. A real number x is computable if there is a base and a computable function that gives the digit at each position of the expansion of x in that base. Equivalently, a real number is computable if there is a computable sequence of rational numbers $(r_n)_{n \geq 0}$ such that $|x - r_n| < 2^{-n}$ for each $n \geq 0$.

Theorem 11 (Turing [12, Theorem 5.1.2])**.** *The following are equivalent:*

(1) *The real x is computable.*
(2) *There is a computable sequence of rationals $(r_n)_{n \geq 0}$ that tends to x such that $|x - r_n| < 2^{-n}$ for all n.*
(3) *There is a computable sequence of rationals $(r_n)_{n \geq 0}$ that converges to x and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $|x - r_{f(n)}| < 2^{-n}$ for all n.*

Theorem 12. *Let $(\lambda_j)_{j \geq 1}$ be computable sequence of integers greater than 2, let $(t_j)_{j \geq 1}$ be a computable sequence of integers monotonically increasing at any speed, and let the starting value a be a rational number. Then, the number α defined by Levin, proved to be absolutely normal in Theorem 1, is computable.*

Proof. The number α is the limit of α_r for r going to infinity, where $a_{\ell_1} = a$ with $\ell_1 = \max(t_1, 2\lceil |\log_2 \log_2 \lambda_1|\rceil + 5)$, and for $r \geq 1$,

$$\alpha_{r+1} = \alpha_r + \frac{a_r}{2^{n_r} q_r},$$

where a_r is an integer in $[0, q_r)$ satisfying the inequalities of Lemma 4, $n_r = 2^r - 2$ and $q_r = 2^{2^r+r+1}$. Lemma 4 proves that such a_r exists. Since $D_{r,j}(c)$ is a computable function it is possible to find a_r by an exhaustive search among all integers in $[0, q_r)$ and all bases λ_j for $j = 1, 2, \ldots, \omega(r)$, where $\omega(r) = 1$ if r in $[1, \ell_2))$, otherwise $\omega(r)$ is the unique index k such that r in $[\ell_k, \ell_{k+1})$, with $\ell_k = \max(t_k, \max\limits_{1 \leq v \leq k} 2\lceil |\log_2 \log_2 \lambda_v|\rceil + 5)$. At each step r, we can compute bitwise approximations of $D_{r,j}$ from above, for each of the possible candidate values of a_r until we find one that satisfies the required inequality for all j between 1 and $\omega(r)$. Thus, the sequence of rationals $\alpha_1, \alpha_2, \ldots$ is computable and converges to an absolutely normal number α. From the proof of Theorem 1 we know that, for each r,

$$|\alpha - \alpha_r| < \frac{2}{2^{n_r}}.$$

Since α is an absolutely normal number, and therefore an irrational number, by Theorem 11 we conclude that α is computable. □

5. The computational complexity of Levin's construction

Theorem 12 proves that under some assumptions of the sequences $(\lambda_j)_{j\geq 1}$ and $(t_j)_{j \geq 1}$, and the starting value a, Levin's construction is indeed an algorithm to compute the number α. The algorithm is recursive. The standard computational model is the Turing machine model, which works just with finite representations, so it only deals with numbers that are the limit of a computable sequence of finite approximations. In this model, at step r, the number of elementary operations needed to find the number a_r cannot be easily determined. This is because to find a_r the algorithm must compute sums of exponential sums. The terms in these sums are transcendental numbers, which can only be computed as limits of finite approximations. It is impossible to determine how many approximations to each term of the exponential sums must be computed to find that a candidate a_r is conclusive. So, instead of counting the number of elementary operations needed to compute the number a_r at step r, here we give the number of mathematical operations needed in an idealized computational model over the real numbers, based on machines with infinite-precision real numbers. A canonical model for this form of computation over the reals is the Blum-Shub-Smale machine [8], abbreviated BSS machine. This is a machine with registers that can store arbitrary real numbers and can compute rational functions over reals at unit cost. Since elementary transcendental functions, as exponential function or trigonometric functions, are not computable by a BSS machine we need to consider the extended BSS machine which includes exponential and trigonometric functions as primitive operations. For our purpose, the extended BSS model is identical to considering Boolean arithmetic circuits augmented with trigonometric functions. Of course, for any given real-valued function, its complexity in the BSS model gives just a lower bound of its complexity in the classical Turing machine model, where the cost for arithmetic (and trigonometric) operations over the real numbers is not constant.

Theorem 13. *Let $(\lambda_j)_{j\geq 1}$ be a computable sequence of reals greater than 1 and let $(t_j)_{j\geq 1}$ be a computable sequence of integers. Levin's algorithm requires*

$$O\left(2^{2^r+3r+1}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right)$$

mathematical operations to compute α_r, for each r.

Proof. Assume a BSS machine which includes exponential and trigonometric functions as primitive operations. The expression $S_{r,j}(m_1, m_2, c)$ is the sum of $\tau_{r,j}$ terms, each of them can be computed in constant time in our machine. Hence, the time needed to compute each value of $S_{r,j}$ is in $O(\tau_{r,j})$. To obtain a value of $D_{r,j}$ we must calculate $O({A_{r,j}}^2) = O(\tau_{r,j})$ values of $S_{r,j}$. Therefore, the computation of $D_{r,j}$ is in $O({\tau_{r,j}}^2) = O((2^r \log_{\lambda_j} 2)^2)$. Finding the value of a_r requires computing $D_{r,j}(c)$ for each j between 1 and $\omega(r)$ until we find a value of c in $[0, q_r)$ which satisfies the inequalities of Lemma 4. In the worst case, it will be necessary to try all possible values for c. In this worst case, the required time is in

$$O\left(\sum_{c=0}^{q_r-1}\sum_{j=1}^{\omega(r)}(2^r\log_{\lambda_j} 2)^2\right)=O\left(q_r\sum_{j=1}^{\omega(r)}(2^r\log_{\lambda_j} 2)^2\right)=O\left(2^{2^r+3r+1}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right).$$

Let T_k be the time required to compute a_k,

$$T_k = 2^{2^k+3k+1}\sum_{j=1}^{\omega(k)}(\log_{\lambda_j} 2)^2$$

Then, the time to compute α_r is $\sum_{k=1}^{r} T_k$. Observe that T_r dominates $\sum_{k=1}^{r-1} T_k$ because

$$\sum_{k=1}^{r-1} T_k \;\leq\; (r-1)T_{r-1} \;=\; (r-1)2^{2^{r-1}+3(r-1)+1}\sum_{j=1}^{\omega(r-1)}(\log_{\lambda_j} 2)^2,$$

and this last expression is in $O\left(2^{2^r+3r+1}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right)$. □

Notice that Theorem 13 estimates the complexity of obtaining a rational approximation α_r with an error bounded by $2^{-(n_{r+1}-1)}$. Since α_r is just an approximation to α, it is not determined how many bits in the expansion of α_r are conclusive so as to conform the expansion of α. One would like that the first $n_{r+1}-1$ bits of α_r determine those of α. As we showed in Proposition 10 Levin's construction is not achievable as the concatenation of the values a_r. An overlapping of the fractions $\frac{a_r}{2^{n_r}q_r}$ may occur, causing carries and changing some of the first bits of α_r.

Lemma 14. *The sum $\sum_{\lambda=2}^{N}(\log_\lambda 2)^2$ has an asymptotic growth in $\Theta\left(\frac{N}{\log N}\right)$.*

Proof. Let $Li(x)$ be the *Eulerian logarithmic integral* [1, Chapter 5], defined as $Li(x) = \int_2^x \frac{dt}{\ln t}$. Then, $\sum_{\lambda=2}^{N}(\log_\lambda 2)^2$ has the same asymptotic growth as $\int_2^N \frac{dt}{(\ln t)^2}$

which is in $\Theta\left(Li(N) - \frac{N}{\log N}\right)$. Since $Li(N)$ is in $\Theta\left(\frac{N}{\log N}\right)$, the lemma is proved. □

Corollary 15. *For $\lambda_j = j+1$, $t_j = 2^j$, $\omega(r) \in O(\log r)$, Levin's algorithm computes a normal number in Borel's sense which requires*

$$O\left(2^{2^r+3r+1}\frac{\log r}{\log\log r}\right)$$

mathematical operations for the r-th approximation α_r.

Theorem 13 proves that the complexity of computing α_r with Levin's original formulation for n_r and q_r, is doubly exponential in r. Since n_r is the number of bits of α_r that are obtained at step r, and in Levin's original formulation n_r is $2^r - 2$, it is fair to say that the complexity of Levin's algorithm is simply exponential in the number of bits computed at step r.

We now prove that, in case n_r is quadratic in r, then Levin's algorithm requires a number of operations that is simply exponential in the square root of the number of bits computed at step r.

Theorem 16. *The variant of Levin's construction with $n_r = r^2$ takes*

$$O\left(r^3 2^{2r}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right)$$

mathematical operations in an extended BSS machine to compute α_r.

Proof. First, we need to choose values for q_r that ensure normality. As we showed in Lemma 7, a sufficient condition is $2^{n_{r+1}-n_r+1+\frac{1}{2}\log(n_{r+1}-n_r+1)} \leq q_r$. We choose

$$q_r = 2^{2r+2+\lceil\log(2r+2)\rceil}.$$

By Theorem 13, to find a_r, in the worst case it is necessary to compute $D_{r,j}(c)$ for each j between 1 and $\omega(r)$ and for each c between 0 and $q_r - 1$ and each $D_{r,j}$ requires $O\left(\tau_{r,j}^2\right)$ operations. Then, the number of operations to find a_r is in $O\left(q_r\sum_{j=1}^{\omega(r)}\tau_{r,j}^2\right)$, because q_r is in $O\left(r2^{2r}\right)$, $\tau_{r,j}$ is in $O\left(r\log_{\lambda_j} 2\right)$, and

$$O\left(q_r\sum_{j=1}^{\omega(r)}\tau_{r,j}^2\right) = O\left(r^3 2^{2r}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right)$$

.

The time to compute α_r is essentially the time required to find a_r because

$$\sum_{k=1}^{r} k^3 2^{2k}\sum_{j=1}^{\omega(k)}(\log_{\lambda_j} 2)^2 \leq r^3 2^{2r+2}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2,$$

which is in $O\left(r^3 2^{2r}\sum_{j=1}^{\omega(r)}(\log_{\lambda_j} 2)^2\right)$. □

Corollary 17. *The variant of Levin's construction with* $n_r = r^2$, $\lambda_j = j+1$, $t_j = 2^j$, $\omega(r) \in O(\log r)$, *takes*

$$O\left(r^3 2^{2r} \frac{\log r}{\log\log r}\right)$$

mathematical operations for the r*-th approximation* α_r.

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