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Abstract. M. Levin defined a real number x that satisfies that the sequence of the fractional parts of $(2^n x)_{n\geq 1}$ are such that the first N terms have discrepancy $O((\log N)^2/N)$, which is the smallest discrepancy known for this kind of parametric sequences. In this work we show that the fractional parts of the sequence $(2^n x)_{n\geq 1}$ fail to have Poissonian pair correlations. Moreover, we show that all the real numbers x that are variants of Levin's number using Pascal triangle matrices are such that the fractional parts of the sequence $(2^n x)_{n\geq 1}$ fail to have Poissonian pair correlations.

Mathematics Subject Classification. 68R15, 11K16, 11K38.

Keywords. Distribution modulo 1, Low discrepancy sequences, Poissonian pair correlations, Borel normal numbers, Pascal triangle matrices, Nested perfect necklaces.

1. Introduction and statement of results. A sequence $(x_n)_{n\geq 1}$ of real numbers in the unit interval is said to have *Poissonian pair correlations* if for all non-negative real numbers s,

$$\lim_{N\to\infty} F_N(s) = 2s$$

where

$$F_N(s) = \frac{1}{N}\#\left\{ (i,j) \,:\, 1 \leq i \neq j \leq N \text{ and } \|x_i - x_j\| < \frac{s}{N} \right\}$$

and $\|x\|$ is the distance between x and its nearest integer. The function $F_N(s)$ counts the number of pairs (x_n, x_m) for $1 \leq m, n \leq N$, $m \neq n$, of points which are within distance at most s/N of each other, in the sense of distance on the torus. If $\lim_{N\to\infty} F(s) = 2s$ for all $s \geq 0$, then the asymptotic distribution of the pair correlations of the sequence is Poissonian, and this explains that the property is referred to as having Poissonian pair correlations. Almost surely a sequence of independent identically distributed random variables in the unit

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interval has this property. Several particular sequences have been proved to have the property, for example, $(\sqrt{n} \mod 1)_{n\geq 1}$ [4]. It is known that for almost all real numbers x $(a_n x \mod 1)_{n\geq 1}$ has the property when a_n is integer valued and $(a_n)_{n\geq 1}$ is lacunary [11]; also when $a_n = n^2$ (or a higher polynomial) [10, 14]. However, for specific values such as $x = \sqrt{2}$ and $a_n = n^2$ it is not known whether $(a_n x \mod 1)_{n\geq 1}$ has Poissonian pair correlations or not.

The property of Poissonian pair correlations implies uniform distribution modulo 1, this was only recently proved in [1, Theorem 1] and also in [5, Corollary 1.2]. The converse does not always hold. Several uniformly distributed sequences of the form $(b^n x \mod 1)_{n\geq 1}$ where b is an integer greater than 1 and x is a constant were proved to fail the property of Poissonian pair correlations. Pirsic and Stockinger [9] proved it for Champernowne's constant (defined in base b). Larcher and Stockinger [6] proved it for x a Stoneham number [12] and for every real number x having an expansion which is an infinite de Bruijn word (see [3, 13] for the presentation of these infinite words). Larcher and Stockinger also show in [7] the failure of the property for other sequences of the form $(a_n x \mod 1)_{n\geq 1}$.

In this paper we show that the sequence $(2^n\lambda \mod 1)_{n\geq 1}$, where λ is the real number defined by Levin in [8, Theorem 2], fails to have Poissonian pair correlations. Levin's number λ is defined constructively using Pascal triangle matrices and satisfies that the discrepancy of the first N terms of the sequence $(2^n\lambda \mod 1)_{n\geq 1}$ is $O((\log N)^2/N)$. This is the smallest discrepancy bound known for sequences of the form $(2^n x \mod 1)_{n\geq 1}$ for some real number x.

We also show that each of the real numbers ρ considered by Becher and Carton in [2] are such that the sequence $(2^n\rho \mod 1)_{n\geq 1}$ fails to have Poissonian pair correlations. These numbers ρ are variants of Levin's number λ because they are defined using rotations of Pascal triangle matrices and the sequence $(2^n\rho \mod 1)_{n\geq 1}$ has the same low discrepancy as that obtained by Levin.

1.1. Levin's number. We start by defining the number λ given by Levin in [8, Theorem 2] and further examined in [2]. As usual, we write $\mathbb{F}_2$ to denote the field of two elements. In this work, we freely make the identification between binary words and vectors on $\mathbb{F}_2$. We define recursively a sequence of matrices on $\mathbb{F}_2$:

$$M_0 = (1) \quad \text{and for every } d \geq 0, \quad M_{d+1} = \begin{pmatrix} M_d & M_d \\ 0 & M_d \end{pmatrix}.$$

The first elements of this sequence, for example, are:

$$M_0 = (1) \quad M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1&1&1&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}.$$

Let d be a non-negative integer and let $e = 2^d$. The matrix $M_d \in \mathbb{F}_2^{e\times e}$ is upper triangular with 1s on the diagonal, hence it is non-singular. Then, if

$$w_0, \ldots, w_{2^e-1}$$

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is the enumeration of all vectors of length e in lexicographical order, the sequence

$$M_d w_0, \ldots, M_d w_{2^e-1}$$

ranges over all vectors of length e. We obtain the d-th block of λ by concatenation of the terms of that sequence:

$$\lambda_d = (M_d w_0)(M_d w_1) \ldots (M_d w_{2^e-1}).$$

Levin's constant λ is defined as the infinite concatenation

$$\lambda = \lambda_0 \lambda_1 \lambda_2 \ldots$$

The expansion of λ in base 2 starts as follows (the spaces are just for convenience):

$$\underbrace{01}_{\lambda_0}\ \underbrace{00\,11\,10\,01}_{\lambda_1}\ \underbrace{0000\,1111\,1010\,0101\,1100\,0011\,0110\,1001\,1000\,0111\,0010\,1101\,0100\,1011\,1110\,0001}_{\lambda_2}$$
$$\underbrace{00000000\,11111111\ldots}_{\lambda_3}$$

Now we introduce a family $\mathcal{L}$ of constants which have similar properties to those of λ. Let σ be the rotation that takes a word and moves its last letter at the beginning: that is, $\sigma(a_1 \ldots a_n) = a_n a_1 \ldots a_{n-1}$. We are going to use σ to define a family of matrices obtained by selectively rotating some of the columns of M_d.

As before, assume d is a non-negative integer and let $e = 2^d$. We say that a tuple $\nu = (n_1, \ldots, n_e)$ of non-negative integers is *suitable* if

$$n_e = 0 \text{ and } n_{i+1} \leq n_i \leq n_{i+1} + 1 \text{ for each } 1 \leq i \leq e-1.$$

Let $C_1, \ldots C_e$ denote the columns of M_d, and let σ^n denote the composition of the rotation σ with itself n times. Then, define

$$M_d^\nu = (\sigma^{n_1}(C_1), \ldots, \sigma^{n_e}(C_e)).$$

For example, by taking $d = 2$ we have 8 different possible matrices, one for every choice of ν:

$$M_2^{(0,0,0,0)} \begin{pmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{pmatrix} \quad M_2^{(1,0,0,0)} \begin{pmatrix} 0&1&1&1\\ 1&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{pmatrix} \quad M_2^{(1,1,0,0)} \begin{pmatrix} 0&0&1&1\\ 1&1&0&1\\ 0&1&1&1\\ 0&0&0&1 \end{pmatrix} \quad M_2^{(2,1,0,0)} \begin{pmatrix} 0&0&1&1\\ 0&1&0&1\\ 1&1&1&1\\ 0&0&0&1 \end{pmatrix}$$

$$M_2^{(1,1,1,0)} \begin{pmatrix} 0&0&0&1\\ 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix} \quad M_2^{(2,1,1,0)} \begin{pmatrix} 0&0&0&1\\ 0&1&1&1\\ 1&1&0&1\\ 0&0&1&1 \end{pmatrix} \quad M_2^{(2,2,1,0)} \begin{pmatrix} 0&0&0&1\\ 0&0&1&1\\ 1&1&0&1\\ 0&1&1&1 \end{pmatrix} \quad M_2^{(3,2,1,0)} \begin{pmatrix} 0&0&0&1\\ 0&0&1&1\\ 0&1&0&1\\ 1&1&1&1 \end{pmatrix}$$

As before, we let $e = 2^d$ and $w_0, \ldots, w_{2^e-1}$ be the increasingly ordered sequence of all vectors in $\mathbb{F}_2^e$. We say that a word is an *e-affine necklace* if it can be written as the concatenation $(Mw'_0)(Mw'_1) \ldots (Mw'_{2^e-1})$ for some $z \in \mathbb{F}_2^e$ and a suitable tuple ν with $M = M_d^\nu$ and $w'_i = w_i + z$ for $0 \leq i \leq 2^e - 1$.

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Finally, we define $\mathcal{L}$ to be the set of all binary words that can be written as an infinite concatenation $\rho_0\rho_1\rho_2\dots$ where every ρ_d is a 2^d-affine necklace. Note that $\lambda \in \mathcal{L}$, by taking $\nu = (0,\dots,0)$ everywhere.

The rest of this note is devoted to proving the following result:

Theorem 1. *For all $\rho \in \mathcal{L}$, the sequence of fractional parts of $(2^n\rho)_{n\geq 1}$ does not have Poissonian pair correlations.*

2. Lemmas. First we prove some necessary results. We present in an alternating manner results about M_d and its corresponding generalizations to the family of matrices M_d^ν.

Lemma 1. *For all d, M_d is triangular and all entries in its diagonal are ones. In particular, M_d is non singular.*

Proof. This is easily proven with induction. $M_0 = (1)$ satisfies the lemma, and if M_d satisfies it, then $M_{d+1} = \begin{pmatrix} M_d & M_d \\ 0 & M_d \end{pmatrix}$ satisfies it too. □

Lemma 2. *For all non-negative d and for every suitable tuple ν, M_d^ν is non-singular.*

Proof. This fact is proven in [2, Lemma 4]. □

Lemma 3. *For all d and for all even n, $M_d w_n$ and $M_d w_{n+1}$ are complementary vectors. That is, the i-th coordinate of $M_d w_n$ equals zero if and only if the i-th coordinate of $M_d w_{n+1}$ equals one.*

Proof. The sequence $w_0, w_2, \dots w_{2^e-1}$ is lexicographically ordered and hence the last entry of w_n is zero whenever n is even. Therefore, w_{n+1} only differs from w_n in the last entry

$$M_d w_{n+1} = M_d w_n + M_d \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = (M_d w_n) + \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

□

To simplify notation, from now on we write $\overline{z}$ for the complementary vector of z. Note that Lemma 3 implies that λ_d can be written as a concatenation of words of the form $w\overline{w}$.

Lemma 4. *For all non-negative d, for all even n, and for every suitable tuple ν, the vectors $M_d^\nu w_n$ and $M_d^\nu w_{n+1}$ are complementary.*

Proof. The last coordinate of ν is zero by definition. Therefore, the last column of M_d^ν is the same as the last column of M_d; that is, it is the vector of ones. The same argument used to prove Lemma 3 applies. □

We say that a vector is *even* if its last entry is 0. Hence, when n is even, w_n is an even vector.

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Lemma 5. *Let d be a non-negative integer and $e = 2^d$. The subspace of all even vectors of length e,*

$$\mathbb{P} = \{v \in \mathbb{F}_2^e \mid v_e = 0\}$$

is invariant under M_d. Furthermore, $M_d w_n$ is an even vector if and only if w_n is an even vector.

Proof. By Lemma 1, M_d is upper triangular and its diagonal is comprised by ones. This implies that all of its columns except the last are even vectors. Therefore, the only way to obtain an odd vector via the computation $M_d w$ is that w itself is odd. □

Lemma 6. *Let d be a non-negative integer and $e = 2^d$. Depending on ν, there are two distinct possibilities:*

(1) *The subspace of even vectors $\mathbb{P}$ is invariant under M_d^ν. In this case, w is an even vector if and only if $M_d^\nu w$ is an even vector.*
(2) *The subspace $\mathbb{P}$ is in bijection with the subspace $\{(v_1, \ldots, v_e) \in \mathbb{F}_2^e \mid v_1 = 0\}$ via M_d^ν. In this case, w is an even vector if and only if $M_d^\nu w$ has a zero in its first coordinate.*

Proof. Take $\nu = (n_1, \ldots n_e)$ such that $n_e = 0$ and $n_{i+1} \leq n_i \leq n_{i+1} + 1$ for all $1 \leq i \leq e - 1$. Combining $n_i \leq n_{i+1} + 1$ for $i = e - 1$ and $n_e = 0$ gives that n_{e-1} is either zero or one. We consider both possibilities separately: the case $n_{e-1} = 0$ will yield (1) and the case $n_{e-1} = 1$ will yield (2).

First, suppose that n_{e-1} equals one. Then, all entries of ν before it must be greater than one. That means that, when building M_d^ν from M_d, all of its columns except the last are rotated at least one position. Fix an index i such that $1 \leq i \leq e - 1$, and consider c the i-th column of the matrix M_d. We show that the first element of $\sigma^{n_i}(c)$ is zero.

By Lemma 1, we know that M_d is triangular. That means that the elements $c_{i+1}, \ldots, c_e$ are necessarily zeros. But the first element of $\sigma^{n_i}(c)$ is c_{e-n_i+1}; and from the inequality $n_i \leq e - i$ it follows that $e - n_i + 1$ is greater or equal than $i + 1$. Therefore, the first element of $\sigma^{n_i}(c)$ is zero.

Because i is any index between 1 and $e - 1$, it follows that the first $e - 1$ columns of M_d^ν have a zero as their first coordinate. If w is an even vector, $M_d w$ is a linear combination of vectors which start with zero, and therefore $M_d w$ also starts with a zero. Conversely, if w begins with a one, then $M_d w$ must be an odd vector, because it's a linear combination of elements that start with a zero and the last column of M_d^ν, which is the vector of ones. We conclude that w is an even vector if and only if $M_d^\nu w$ starts with a zero.

The case where n_{e-1} equals zero is analogous, and we give an outline of the proof. First, prove that the first $e - 1$ columns of M_d^ν are even vectors. Then, $M_d^\nu w$ is even if and only if w is even. □

3. Proof of the main theorem. We first prove the theorem for λ and at the end we explain how to generalize the result to each number in the family $\mathcal{L}$. For any given non-negative d, we set $e = 2^d$ and show that for an appropriate choice of increasing N which depends on d and e, $F_N(2)$ diverges. For this

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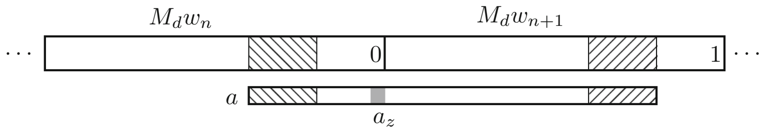

FIGURE 1. An occurrence of a

we show that some selected patterns have too many occurrences in λ_d. More precisely, we count occurrences of binary words of length $d + e$,

$$a = a_1 a_2 \dots a_{d+e}$$

such that

$$\overline{a_1 \dots a_d} = a_{e+1} \dots a_{e+d}.$$

The reason for this choice comes from Lemma 3 and it will soon become clear. We need some terminology. Given a word a as above and an occurrence of a in λ_d,

(1) let k be the number of zeros in $a_d \dots a_e$;
(2) let n be the index such that the a occurs in $M_d w_n$;
(3) let z be the position in a that matches with the e-th (that is the last) symbol of $M_d w_n$ (Fig. 1).

We require n to be an even number and z to be in the range $d \leq z \leq e$. The latter is to prevent the word a from spanning over more than two words, and the former is to ensure that a match for the first d letters automatically yields a match for the last d letters (a combination of Lemma 3 and the hypothesis over a). In addition, by Lemma 5 we know that $M_d w_n$ is an even word, and therefore a_z must be zero.

We fix k and count all possible occurrences in λ_d of every possible word a. There are exactly

$$2^{d-1} \binom{e-d+1}{k}$$

words a with k zeros in $a_d \dots a_e$. For every one of them, we have a choice of k different z, because we know that a_z must be zero. We claim that each of those choices for z correspond to an actual occurrence of a in λ_d. Let us suppose that the binary word a, whose length is $d + e$, starts in $M_d w_n$ and continues in $M_d w_{n+1}$. Then, it must hold that, for some z (Fig. 2),

$$a_1 \dots a_z = (M_d w_n)_{e-z+1} \dots (M_d w_n)_e$$

and

$$a_{z+1} \dots a_e = (M_d w_{n+1})_1 \dots (M_d w_{n+1})_{e-z}.$$

So, Lemma 3 allows us to conclude that

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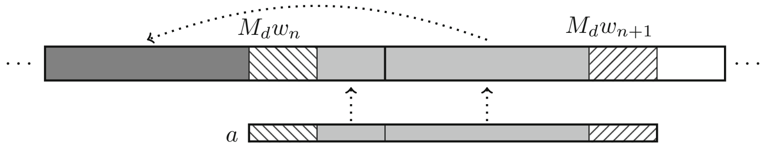

FIGURE 2. Given an a and a choice of z, it is possible to find an unique position within λ_d where the word a occurs with alignment z

$$M_d w_n = \overline{a_{z+1} \dots a_e} a_1 \dots a_z.$$

By Lemma 1, we know that there exists some w_n that satisfies this equation and by Lemma 5, n must be an even number. Therefore, given a choice of z, there is an occurrence of a. We conclude that for every choice of k, and for every word $a \in \{0,1\}^{e+d}$ with k zeros in $a_d \dots a_e$, we have exactly k occurrences within λ_d.

We now prove that the sequence of fractional parts of $(2^n\lambda)_{n\geq 1}$ does not have Poissonian pair correlations. Take $s = 2$ and $N = 2^{d+e+1}$. We prove that $\lim_{d\to\infty} F_N(s) = \infty$. In order to do that, we note that two different occurrences of the same word $a \in \{0,1\}^{e+d}$ correspond to two different suffixes of λ that share its first $e + d$ digits.

We write $\{x\}$ to denote $x - \lfloor x \rfloor$, the fractional expansion of x. If a has two different occurrences within λ at positions i and j, then

$$\begin{aligned}
\| \{2^i\lambda\} - \{2^j\lambda\} \| &= \|0.a_1 \dots a_{e+d}\lambda_{i+d+e+1} \dots - 0.a_1 \dots a_{e+d}\lambda_{j+d+e+1} \dots \| \\
&\leq |0.a_1 \dots a_{e+d}\lambda_{i+d+e+1} \dots - 0.a_1 \dots a_{e+d}\lambda_{j+d+e+1} \dots | \\
&< 2^{-(e+d)} \\
&= \frac{s}{N}.
\end{aligned}$$

Therefore, if i and j are both no greater than N, the pairs (i, j) and (j, i) count for $F_N(s)$. For indices i and j of λ which correspond to elements of λ_d this is the case:

$$|\lambda_0 \dots \lambda_d| = \sum_{i=0}^{d} |\lambda_i| = \sum_{i=0}^{d} 2^i 2^{2^i} = \sum_{i=0}^{d} 2^{i+2^i} < 2^{2^d+d+1} = N.$$

We are now able to give a lower bound for $F_N(s)$. To do this, we count all possible pairs of occurrences of every word a satisfying the condition $\overline{a_1 \dots a_d} = a_{e+1} \dots a_{e+d}$ in λ_d. Recall that a word a with k zeros in the middle has exactly k occurrences in λ_d, so by summing over k we get:

$$F_N(2) \geq \frac{1}{N} \sum_{k=0}^{e-d+1} 2 \left(2^{d-1} \binom{e-d+1}{k} \binom{k}{2} \right)$$

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$$
\begin{aligned}
&= \frac{1}{2^{e+1}} \sum_{k=0}^{e-d+1} \binom{e-d+1}{k}\binom{k}{2} \\
&= \frac{1}{2^{e+1}} \binom{e-d+1}{2} 2^{e-d+1-2} \\
&= \frac{1}{8e}(e-d+1)(e-d).
\end{aligned}
$$

In the third step, we applied the identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{2} = 2^{n-2}\binom{n}{2}$$

with $n = (e-d+1)$. Thus,

$$F_N(2) \geq \frac{(e-d+1)(e-d)}{8e}$$

and the last expression diverges as $d \to \infty$ because e is squared in the numerator but linear in the denominator and d is insignificant with respect to e. This concludes the proof that the sequence of the fractional parts of $(2^n\lambda)_{n\geq 1}$ does not have Poissonian pair correlations.

We now explain how to extend the proof for any given constant in $\mathcal{L}$. Take $\rho \in \mathcal{L}$. Then ρ can be written as a concatenation

$$\rho = \rho_0\rho_1\rho_2\ldots$$

where each ρ_d is a 2^d-affine necklace. That means that for every d, there exists a suitable tuple ν such that

$$\rho_d = (M_d^\nu w_0)(M_d^\nu w_1)\ldots(M_d^\nu w_{2^e-1}).$$

We take $s = 2$ and $N_d = 2^{e+d+1}$ and we prove that the sequence $F_{N_d}(s)$ diverges as $d \to \infty$. As we did for λ, it is possible to give a lower bound for $F_{N_d}(s)$ by counting occurrences within ρ_d of words of length $e+d$.

Fix a non-negative integer d. By Lemma 6, there are two possibilities: either M_d^ν maps the subspace of even vectors $\mathbb{P}$ to itself or it maps it to the set of vectors beginning with zero. In the first case, we can replicate essentially verbatim the procedure we followed for λ to get a lower bound for $F_{N_d}(s)$. In the second case, we have to slightly alter the argument: z is redefined to be the index of a such that a_z matches the first letter of $(M_d^\nu w_{n+1})$, and k is redefined to be the number of ones in $a_{d+1}\ldots a_{e+1}$. Despite these modifications, we reach the same lower bound for $F_{N_d}(s)$. Since it diverges as $d \to \infty$ we conclude that the sequence of the fractional parts of $(2^n\rho)_{n\geq 1}$ does not have Poissonian pair correlations.

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