Normal numbers in sparse Cantor sets

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Abstract

We consider Cantor-type sets of Hausdorff dimension zero, consisting of all numbers whose base-2 expansion can have a 1 only at positions belonging to a given sparse set (local count at least $\log k$ in every interval of length k). We prove that the measure induced by independent, non-identically distributed Bernoulli digits assigns full mass to numbers that are normal in every odd base. The proof extends Schmidt's 1960 method to this Hausdorff zero-dimensional setting, and we provide an explicit algorithmic construction of such numbers—yielding the first known examples of numbers deterministic in base 2 yet normal in all odd bases. This work supports our broader conjecture that given determinism in one base, normality in all multiplicatively independent bases is prevalent.

Contents

1 Introduction and Statement of Results

A base is an integer greater than or equal to 2. For a real number x, its expansion in base b is the sequence of digits $b_1(x), b_2(x), \dots$ such that

$$x = \lfloor x \rfloor + \sum_{j \geq 1} b_j(x) b^{-j}, \qquad b_j(x) \in \{0, \dots b-1\}.$$

where we assume the expansion does not end with a tail of $b-1$s. A real number x is simply normal in base b if each digit $\{0, \dots, b-1\}$ occurs with asymptotic frequency $1/b$ in the expansion of x in base b. A real number x is normal in base b if it is simply normal in base b^k, for every $k \geq 1$. For a presentation of normal numbers see [9, 14].

We write log for the natural logarithm. For integers a, c we write $[a, c]$ for the set of integers between a and c inclusive. For a set $\mathcal{S}$ of positive integers let $S(a, c) = \#(\mathcal{S} \cap [a, c])$.

Definition (Sparse set)**.** A set $\mathcal{S}$ of positive integers is sparse if it has density zero and there exists a sparsity exponent $\rho > 0$, such that

$$\liminf_{k\to\infty} \min_{0\leq a\leq k^\rho} \frac{S(a, a+k)}{30\log k} > 1.$$

The prime numbers, the triangular numbers, and the squares are all examples of sparse sets. Also $\{\lceil e^{j/100} \rceil : j \geq 1\}$ is sparse. However, neither the powers of two, nor the factorials are sparse sets.

Definition (Sparse Cantor set $\mathcal{C}(\mathcal{S})$)**.** For a sparse set $\mathcal{S}$ of positive integers, the sparse Cantor set $\mathcal{C}(\mathcal{S})$ is the set of real numbers in the unit interval whose binary digits can be 1 only at positions belonging to the sparse set $\mathcal{S}$. That is,

$$\mathcal{C}(\mathcal{S}) = \Big\{x \in [0, 1) : x = \sum_{k\in\mathcal{S}} b_k 2^{-k}\Big\}, \quad (b_k \in \{0, 1\})$$

We prove the following three results.

Theorem 1. *Let $\mathcal{S}$ be a sparse set of positive integers. There are uncountably many numbers in $\mathcal{C}(\mathcal{S})$ that are normal in every odd base.*

Theorem 2. *Let $\mathcal{S}$ be a sparse set of positive integers. For any even base b, there are uncountably many numbers in $\mathcal{C}(\mathcal{S})$ that are not normal in any even base $\leq b$.*

A function from non-negative integers to non-negative integers is computable if it belongs to the smallest class of functions containing the constant functions, successor, projections and it is closed under composition, primitive recursion, and unbounded minimization. A set of positive integers is computable if its characteristic function is a computable function defined for all positive integers. A real number is computable if there is a computable function that, for each n, gives the n-th digit of its expansion in some prescribed base. The computable functions are realized by algorithms.

Since the proof of Theorem 2 is constructive, given any computable set $\mathcal{S}$ with a computable sparsity exponent, the proof immediately yields a distinct computable number for each computable increasing sequence of elements in $\mathcal{S}$ with exponential growth. In contrast, the proof of Theorem 1 is nonconstructive; we therefore provide an explicit algorithm.

Theorem 3. *Let $\mathcal{S}$ be a computable sparse set of positive integers with a computable sparsity exponent. There is an algorithm that given a rational ε, $0 < \varepsilon < 2/3$, produces a number $x(\varepsilon)$ in $\mathcal{C}(\mathcal{S})$ expressed in base two, that is normal in all odd bases.*

The set $\mathcal{C}(\mathcal{S})$ is generated by a non-stationary iterated function system consisting of the two similarity maps

$$f_0(x) = \frac{x}{2}, \qquad f_1(x) = \frac{x}{2} + \frac{1}{2}, \qquad x \in [0, 1].$$

However, unlike a classical self-similar set, the available maps depend on the digit position k: at step k, the map f_1 is allowed if and only if $k \in \mathcal{S}$; if $k \notin \mathcal{S}$, then only the map f_0 can be used. Thus, $\mathcal{C}(\mathcal{S})$ is not a self-similar set generated by a single fixed IFS applied uniformly at every scale, so our theorem does not follow from the sufficient condition for normality given by the geometrical criterion of Hochman and Shmerkin [13] for self-similar measures. It does not follow either from the extension done by Algom, Rodriguez Hertz, and Wang [1] that treat stationary self-conformal measures and prove, under aperiodicity of the derivative cocycle, that such measures are pointwise normal in all integer bases and Rajchman. Their result relies on the existence of a fixed IFS of conformal maps. Measures supported on sets of Hausdorff dimension zero are not guaranteed to have a Fourier transform that decays sufficiently fast at infinity to ensure equidistribution.

We consider Cantor-type sets $\mathcal{C}(\mathcal{S})$ consisting of all numbers whose base-2 expansion can have a 1 only at positions belonging to a sparse set $\mathcal{S}$. Theorem 1 establishes that the measure on $\mathcal{C}(\mathcal{S})$ induced by independent, non-identically distributed Bernoulli digits—fair on the sparse set and zero outside it—is supported on numbers that are normal in all odd bases. Theorem 1 yields a novel case of normality in a setting where the measure is not self-conformal and the underlying set $\mathcal{C}(\mathcal{S})$ has Hausdorff dimension zero. An analogous situation occurs with Liouville numbers: although the set of Liouville numbers has Hausdorff dimension zero, almost all numbers in the support of Bluhm's measure, which consists entirely of Liouville numbers, are normal in every base [8, 5].

The set $\mathcal{C}(\mathcal{S})$ is made of just two digits, being 0 everywhere except at sparse positions, where there is a choice between 0 and 1. This is the smallest amount of randomness for which we could prove that the Fourier transform decays fast enough to yield normality. Theorem 1 asserts normality only to odd bases, keeping the proof simpler than if we had to account for multiplicative independent bases. Applying Schmidt's specialized tools from [18] it can be extended to all bases multiplicatively independent of 2. Besides, the proof of Theorem 1 generalizes in a routine way to Cantor-type sets constructed from b digits for $b \geq 3$, yielding non-normality to base b and normality to all bases co-prime with b.

Theorem 1 generalizes the singular case of Volkmann's theorem in [20], where he deals with a fixed set $\mathcal{S}$ of triangular numbers and considers a Cantor-type set $\mathcal{C}(\mathcal{S})$ with three digits. The numbers in $\mathcal{C}(\mathcal{S})$ have either of two prescribed digits at the triangular positions, while at the non-triangular positions the third digit is fixed. For any base $b \geq 3$, Volkmann proves that uncountably many numbers in $\mathcal{C}(\mathcal{S})$ are normal in all bases multiplicatively independent of b. In contrast to Volkmann's work, Theorem 1 applies not just when $\mathcal{S}$ is the set of triangular numbers, but to any sparse set whose gaps grow at most exponentially (the nth element of $\mathcal{S}$ is $O(e^{\rho n})$), with a uniform proof over all such $\mathcal{S}$.

Both, Volkmann's proof [20] and our proof of Theorem 1, build on Schmidt's work in [18], which itself generalizes Cassels's [11] (see also [6, 3]). Schmidt gives an explicit construction of uncountably many numbers that are normal in all bases in a set X and not normal in any base in a set Y, where X and Y are closed under multiplicative dependence. For each base $b \geq 3$, his construction uses Cantor-type sets with two digits, of Hausdorff dimension $(\log 2)/(\log b)$. In contrast, Volkmann's proof and our proof of Theorem 1 use Cantor-type sets of Hausdorff dimension zero. To obtain our result uniformly over all sparse sets $\mathcal{S}$, we redevelop Schmidt's [18, Hilfssatz 5], a key lemma that bounds the sum of a Riesz product to yield the needed exponential sum estimate; Volkmann's proof does not require this.

Theorem 2 proves a result independent of that of Theorem 1 concerning the failure of normality in even bases. The proof is constructive. We do not know how to prove a similar result for lack of normality in $\mathcal{C}(\mathcal{S})$ for odd bases. Topologically, fluctuating behaviour of the frequency of digits or blocks of digits is the most common form of non-normality in the full unit interval: in any given base, the set of real numbers exhibiting it is comeager in the full unit interval [16, 2]. However, in the Cantor-type set $\mathcal{C}(\mathcal{S})$ the usual topological arguments do not apply.

In Theorem 3 we give an algorithm that, given a sparse set $\mathcal{S}$ and a positive rational $\varepsilon < 2/3$, produces a number $x(\varepsilon)$ in $\mathcal{C}(\mathcal{S})$ that is normal in all odd bases. This answers a question of Pablo Shmerkin (personal communication, 2014), who asked for numbers whose digit and block frequencies converge to non-uniform limits in at least one base, yet are normal in the other multiplicatively independent bases. We remark that Volkmann's work [20] does not provide a computable construction in his setting. To our knowledge, Theorem 3 is the first algorithm that yields a number that has digit 0 with asymptotic frequency 1 in its base-2 expansion and is normal in all odd bases. Our algorithm can be generalized to any choice of base in place of 2 and any prescribed non-uniform asymptotic frequencies for digits or blocks of digits in that base, while preserving normality in all multiplicatively independent bases.

We base our algorithm in the computable reformulation of Sierpiński's construction [19] of an absolutely normal number done by the first author in [4]. We adapt it in two aspects. First, we define the set of numbers in $\mathcal{C}(\mathcal{S})$ whose initial segment deviates from normality in terms of averages of exponential sums, whereas in [4] such sets are unions of sets satisfying a discrete condition (digit occurrence counts). Second, the algorithm is adapted to make choices only at steps k where k is in the sparse set $\mathcal{S}$. The resulting algorithm outputs the base-2 expansion of a real number x having 0s at all positions not in $\mathcal{S}$. For the elements of $\mathcal{S}$ taken in increasing order, at the i-th element the algorithm demands double exponentially in i many steps to determine whether to output a 0 or a 1. The main obstacle to proving Theorem 3 with a direct adaptation of Schmidt's construction [18]—which could possibly yield a polynomial-time algorithm—lies in controlling the multiplicities of the residue classes $\lfloor \ell r^n/2^a \rfloor \bmod 2^k$ for fixed positive integers a, k, odd ℓ, r, as n ranges over $0, \dots, 2^k - 1$.

Regardless of the choice of the sparse set $\mathcal{S}$, all numbers in $\mathcal{C}(\mathcal{S})$ have a base-2 expansion in which the digit 1 occurs with asymptotic density 0, hence the digit 0 occurs with asymptotic density 1. Such numbers, which are of course not normal in base 2, form a subclass of the deterministic numbers in base 2. The notion of deterministic numbers was introduced by Weiss [21] in the setting of dynamical systems: a number is deterministic in base b if the orbit of its base-b expansion under the shift map generates empirical measures (for each n, the average of point masses on the first n shifted sequences $x, Tx, \dots, T^{n-1}x$) whose weak-$*$ limits all have zero Kolmogorov–Sinai entropy. Zero entropy means that the statistical description of the orbit never exhibits exponential growth in the number of distinguishable finite patterns. Deterministic numbers admit an equivalent characterization due to Rauzy [17]: a real number is deterministic in base b if and only if adding it to any base-b normal number x preserves normality—that is, $x + y$ remains normal in base b. This builds on a preservation result by Spears and Maxfield [15], who proved that adding a number whose base-b expansion contains a single digit with asymptotic density 1 preserves normality. Besides numbers with base-2 expansion with asymptotic density of 0s equal to 1, examples of deterministic numbers in base 2 include numbers with periodic base-2 expansions, Sturmian numbers, and numbers whose base-2 expansion is a paper-folding sequence. Theorems 1 and 2 yield the following.

Corollary 1. *There are uncountably many deterministic numbers in base 2 that are normal in all odd bases; and for any even base b, uncountably many that are not normal in all even bases $\leq b$ simultaneously. In both cases, we give an explicit construction of a countable family of such numbers.*

Bernay [7] proved that the set of deterministic numbers has Hausdorff dimension 0. We conjecture that, among the deterministic numbers in base 2, the normality in all bases multiplicatively independent of 2 is a generic property.

2 Proof of Theorem 1

Definition (Measure μ on $\mathcal{C}(\mathcal{S})$)**.** Given a sparse set $\mathcal{S}$ of positive integers, we define a probability measure $\mu = \mu(\mathcal{S})$ on the unit interval $[0, 1)$ equipped with the Borel σ-algebra, via the binary expansion of numbers. The numbers $x \in \mathcal{C}(\mathcal{S})$ are of the form $x = \sum_{k\in\mathcal{S}} b_k 2^{-k}$, where, for every $k \geq 1$, digit $b_k = 0$ with probability $1/2$ if $k \in \mathcal{S}$, and digit $b_k = 0$ with probability 1 otherwise.

So μ is the image of a product of independent but not identically distributed Bernoulli measures under the binary expansion map. The measure μ is singular and continuous with respect to Lebesgue measure, and its support is the set $\mathcal{C}(\mathcal{S})$. The Fourier transform of μ is

$$\widehat{\mu}(h) = \prod_{k\in\mathcal{S}} e^{-\pi i h/2^k} \cos(\pi h/2^k) = e^{-\left(\pi i h \sum\limits_{k\in\mathcal{S}} 2^{-k}\right)} \prod_{k\in\mathcal{S}} \cos(\pi h/2^k).$$

So,

$$|\widehat{\mu}(h)| \leq \prod_{k\in\mathcal{S}} \left|\cos(\pi h/2^k)\right|.$$

We show that for any sparse set $\mathcal{S}$, the measure μ is a Rajchman measure whose decay is rapid enough to satisfy the Davenport–Erdős–LeVeque theorem [12] for all odd bases. It follows that μ-almost every $x \in \mathcal{C}(\mathcal{S})$ is normal in all odd bases. This is our strategy to prove Theorem 1.

Lemma 1. *Let $\mathcal{S}$ be a sparse set and let ρ be its sparsity exponent. Then for all integers $a \geq 0$ and all integers $k \geq \delta_1(a)$,*

$$S(a, a + k) > 30 \log k,$$

where $\delta_1(a) = \big(a^{1/\rho} + 1\big)K$, with $K \geq 2$ a constant depending only on $\mathcal{S}$.

Proof. From the definition of a sparse set we can choose an integer K_0 such that

$$S(a, a + k) > 30 \log k \qquad \text{for all } k \geq K_0 \text{ and } 0 \leq a \leq k^\rho.$$

Set $K = \max\{K_0, 2\}$, so that $\log k > 0$ for $k \geq K$. Fix any $a \geq 0$ and any $k \geq (a^{1/\rho} + 1)K$. Then $k \geq K \geq K_0$ and $k > a^{1/\rho}$, whence $k^\rho > a$ and, because a is an integer, $a \leq k^\rho$. The required inequality follows. □

For an integer x we write $\nu_2(x)$ for the exponent of the highest power of 2 dividing x.

Lemma 2. *Let integers $k \geq 1$, $r \geq 3$ odd and ℓ odd. Among the 2^k numbers*

$$\ell, \ell r, \ell r^2, \ldots, \ell r^{2^k-1}$$

at most δ_2 lie in the same residue class modulo 2^k, where $\delta_2 = 2^{\nu_2(r^2-1)-1}$.

Proof. Let $h = \nu_2(r^2-1)$ For $k \geq 1$, let $d_k = \mathrm{ord}_{2^k}(r)$ be the multiplicative order of r modulo 2^k. Since $(\mathbb{Z}/2^k\mathbb{Z})^\times$ is a 2-group, d_k is a power of 2, say $d_k = 2^m$, with $m < k$. We bound m from below. Since $r^{2^m} \equiv 1 \pmod{2^k}$, we have $2^k \mid r^{2^m} - 1$. Using the Lifting-The-Exponent lemma we show by induction that $\nu_2(r^{2^t}-1) = h+t-1$ for $t \geq 1$. The case $t = 1$ is the definition of h. For $t \geq 2$, write $r^{2^t} - 1 = (r^{2^{t-1}}-1)(r^{2^{t-1}}+1)$. Since $r^{2^{t-1}} \equiv 1 \pmod 4$, $\nu_2(r^{2^{t-1}}+1) = 1$ and by the inductive hypothesis $\nu_2(r^{2^{t-1}}-1) = h+t-2$. So

$$\nu_2(r^{2^t}-1) = \nu_2(r^{2^{t-1}}-1) + \nu_2(r^{2^{t-1}}+1) = h+t-1.$$

With $t = m$, we get $h+m-1 \geq k$, hence $m \geq k-h+1$ and $d_k \geq 2^{k-h+1}$. For a fixed residue A modulo 2^k, the number of $n \in \{0, \ldots, 2^k-1\}$ with $\ell r^n \equiv A \pmod{2^k}$ is either 0 or exactly $2^k/d_k$, because $n \mapsto r^n$ is a homomorphism $\mathbb{Z} \to (\mathbb{Z}/2^k\mathbb{Z})^\times$ with kernel $d_k\mathbb{Z}$. Thus, the maximum number of n yielding the same residue is at most $2^k/d_k \leq 2^{h-1}$. Hence, $\delta_2 = 2^{h-1} = 2^{\nu_2(r^2-1)-1}$. □

By digits in base 2 we mean the numbers $0, 1$. For an integer $q \geq 1$, each integer x such that $0 \leq x < 2^q$ has a q-bit binary representation $x = \sum_{i=1}^q c_i 2^{i-1}$ with $c_i \in \{0,1\}$. The digit pairs of x (for this fixed q) are the pairs $(c_q, c_{q-1}), \ldots, (c_2, c_1)$. For a non-negative integer a, an *a-obedient digit pair* is a pair of digits $c_{i-a}c_{i-1-a}$ that are not both equal to 0 nor both equal to 1, and for which the index i is sparse. We let $z_a(x)$ denote the number of a-obedient digit pairs $c_{i-a}c_{i-1-a}$ of x.

Lemma 3. *Let $\mathcal{S}$ be a sparse set. For all integers $a \geq 0$ and all integers $q \geq \delta_3(a)$,*

$$\#\Big\{0 \leq x < 2^q : z_a(x) < \Big\lfloor 2.9 \log q \Big\rfloor\Big\} < 2^q q^{-2}$$

where $\delta_3(a) = \max(\delta_1(a), 10^{30})$ with $\delta_1(a)$ determined in Lemma 1.

Proof. Lemma 1 gives $S(a, a+q) > 30 \log q$ for $q \geq \delta_1(a)$. Choose a maximum subset $\mathcal{I} \subseteq (\mathcal{S} \cap [a+2, a+q])$ with no two consecutive elements. Let $I = \#\mathcal{I}$. Then,

$$I \geq \frac{1}{2}\big(S(a, a+q) - 2\big) > 15 \log q - 1.$$

Since any two distinct elements of $\mathcal{I}$ differ by at least 2, the first coordinates of the a-obedient digit pairs $(i-a, i-1-a)$, with $i \in \mathcal{I}$, are necessarily pairwise distinct. Set $L = \lfloor \frac{6}{31} I \rfloor$. Since $L \leq I/2$, the binomial tail bound yields

$$\#\{0 \leq x < 2^q : z_a(x) < L\} \leq 2^{q-I} \sum_{t=0}^{L-1} \binom{I}{t} \leq L \cdot 2^{q-I} \binom{I}{L}.$$

Using the entropy inequality $\binom{n}{k} \leq 2^{nH(k/n)}$ and $H(6/31) < 0.709$ (so $1 - H(6/31) > 0.291$),

$$L \cdot 2^{q-0.291I} \leq q \cdot 2^{q-0.291I}.$$

Since $I > 15\log q - 1$, $0.291I > 4.365\log q - 0.291$. For $q \geq 10^5$,

$$4.365\log q - 0.291 > 3\log_2 q = \frac{3}{\ln 2}\log q \approx 4.328\log q,$$

so $0.291I > 3\log_2 q$ and $2^{-0.291I} < q^{-3}$. Consequently

$$\#\{0 \leq x < 2^q : z_a(x) < L\} < q \cdot 2^q \cdot q^{-3} = 2^q q^{-2}.$$

Finally, for all $q \geq 10^{30}$,

$$\frac{6}{31}I > \frac{90}{31}\log q - \frac{6}{31} \geq 2.9\log q.$$

Hence $L = \left\lfloor \frac{6}{31}I \right\rfloor \geq \lfloor 2.9\log q \rfloor$. □

Lemma 4. *Let $\mathcal{S}$ be a sparse set, let $a \geq 0$ be an integer and let $r \geq 3$ and $\ell > 0$ be odd integers. Then, for every $N \geq \delta_4(a)$,*

$$\#\Big\{0 \leq n < N : z_a(\ell r^n) < \big\lfloor 2.9\log\lfloor \log_2 N \rfloor \big\rfloor \Big\} \leq (\log 2)^2\, 2^{\nu_2(r^2-1)+2}\frac{N}{(\log N)^2},$$

where $\delta_4(a) = 2^{\delta_3(a)}$ with $\delta_3(a)$ determined in Lemma 3.

Proof. Set $q = \lfloor \log_2 N \rfloor$ and $L = \big\lfloor 2.9\log q \big\rfloor$. For all $N \geq \delta_4(a)$ we have $q \geq \delta_3(a)$. Applying Lemma 3 gives

$$\#\{\, 0 < x \leq 2^q : z_a(x) < L \,\} < 2^q q^{-2}.$$

For each n, $0 \leq n < N$, let $Y_n = \ell r^n \bmod 2^q$. Then $z_a(\ell r^n) \geq z_a(Y_n)$ and $Y_n \neq 0$ because ℓ, r are odd. The sequence $(Y_n)_{n\geq 0}$ is periodic, and its period divides 2^q. Therefore, as n ranges from 0 to $N-1$, the sequence covers at most two full periods of length 2^q; that is, at most $2 \cdot 2^q$ terms. By Lemma 2 each residue class modulo 2^q occurs at most $\delta_2 = 2^{\nu_2(r^2-1)-1}$ times so for every x,

$$\#\{0 \leq n < N : Y_n = x\} \leq 2\delta_2.$$

Consequently,

$$\#\{0 \leq n < N : z_a(\ell r^n) < L\} \leq \sum_{\substack{x=0 \\ z_a(x)<L}}^{2^q-1} 2\delta_2 \leq 2\delta_2 \cdot 2^q q^{-2} \;\leq\; 2\delta_2\, N\, q^{-2}.$$

Now $q \geq \frac{\log N}{2\log 2}$ for every $N \geq 4$, so $q^{-2} \leq (2\log 2)^2(\log N)^{-2}$. Thus

$$\#\{n < N : z_a(\ell r^n) < L\} \leq 2(2\log 2)^2\delta_2\frac{N}{(\log N)^2}.$$

Since $L = \big\lfloor 2.9\log q \big\rfloor$ and $q = \lfloor \log_2 N \rfloor$ the lemma is proved. □

Lemma 5. *Let $\mathcal{S}$ be a sparse set, let $a \geq 0$ be an integer and let $r \geq 3$ and $\ell > 0$ be odd integers. For every $N \geq \delta_4(a)$,*

$$\sum_{n=0}^{N-1} \prod_{\substack{k>a \\ k\in\mathcal{S}}} \left|\cos(\pi\ell r^n/2^{k-a})\right| \leq 2^{\nu_2(r^2-1)+2}\,\frac{N}{(\log N)^{1.005}}.$$

with $\delta_4(a)$ determined in Lemma 4 .

Proof. Set $q = \lfloor \log_2 N \rfloor$ and $L = \lfloor 2.9 \log q \rfloor$. Split $\{0, \ldots, N-1\}$ into

$$\mathcal{A} = \{0 \leq n < N : z_a(\ell r^n) < L\}, \qquad \mathcal{B} = \{0 \leq n < N : z_a(\ell r^n) \geq L\}.$$

Contribution of $\mathcal{A}$. Each factor in the product is ≤ 1. By Lemma 4 for all $N \geq \delta_4(a)$,

$$\#\mathcal{A} \leq (\log 2)^2\, 2^{\nu_2(r^2-1)+2}\frac{N}{(\log N)^2},$$

Hence,

$$\sum_{n\in\mathcal{A}} \prod_{\substack{k>a \\ k\in\mathcal{S}}} |\cos(\pi\ell r^n/2^{k-a})| \leq K_{\mathcal{A}}\frac{N}{(\log N)^2}, \qquad \text{with } K_{\mathcal{A}} = (\log 2)^2\, 2^{\nu_2(r^2-1)+2}.$$

Contribution of $\mathcal{B}$. For $n \in \mathcal{B}$ write $x = \ell r^n = \sum_{j\geq 1} c_j 2^{j-1}$ ($c_j \in \{0,1\}$). For any sparse index $i > a$ with $i \leq a+q$, the two highest bits of $x \bmod 2^{i-a}$ are c_{i-a} and c_{i-a-1}. If the pair is obedient, the fractional part of $x/2^{i-a}$ lies in $[\frac{1}{4}, \frac{3}{4})$, where $|\cos(\pi x/2^{i-a})| \leq 1/\sqrt{2}$. By definition, $z_a(x)$ counts exactly such indices i with $a+2 \leq i \leq a+q$. The indices $i > a+q$ and $a < i < a+2$ contribute at most a factor 1. Since $n \in \mathcal{B}$, we have $z_a(x) \geq L$, and therefore

$$\prod_{\substack{k>a \\ k\in\mathcal{S}}} |\cos(\pi x/2^{k-a})| \leq \Big(\frac{1}{\sqrt{2}}\Big)^L = 2^{-L/2}.$$

Now $L = \lfloor 2.9 \log q \rfloor \geq 2.9 \log q - 1$, so

$$2^{-L/2} \leq 2^{-(2.9\log q - 1)/2} = \sqrt{2}\, q^{-(2.9/2)\ln 2}.$$

Since $q = \lfloor \log_2 N \rfloor \geq \frac{\log N}{2\log 2}$ for $N \geq 4$, we obtain

$$\sqrt{2} q^{-\frac{2.9}{2}\ln 2} \leq \sqrt{2}(2\log 2)^{\frac{2.9}{2}\ln 2}\,(\log N)^{-\frac{2.9}{2}\ln 2} \leq K_{\mathcal{B}}\,(\log N)^{-1.005},$$

with $K_{\mathcal{B}} = \sqrt{2}(2\log 2)^{\frac{2.9}{2}\ln 2}$. Thus, for every $n \in \mathcal{B}$,

$$\prod_{\substack{k>a \\ k\in\mathcal{S}}} |\cos(\pi\ell r^n/2^{k-a})| \leq K_{\mathcal{B}}\,(\log N)^{-1.005}.$$

Consequently,

$$\sum_{n\in\mathcal{B}} \prod_{\substack{k>a \\ k\in\mathcal{S}}} |\cos(\pi\ell r^n/2^{k-a})| \leq K_{\mathcal{B}}\,\frac{N}{(\log N)^{1.005}}.$$

Combining the contributions of $\mathcal{A}$ and $\mathcal{B}$. For $N \geq N_1 = \max(N_0, 4)$,

$$\sum_{n=0}^{N-1} \prod_{\substack{k>a \\ k\in\mathcal{S}}} |\cos(\pi\ell r^n/2^{k-a})| \leq K_{\mathcal{A}} \frac{N}{(\log N)^2} + K_{\mathcal{B}} \frac{N}{(\log N)^{1.005}}.$$

For $N \geq 3$ we have $(\log N)^{-2} \leq (\log N)^{-1.005}$. Hence, the whole sum is bounded by $(K_{\mathcal{A}} + K_{\mathcal{B}})N/(\log N)^{1.005} \leq 2^{\nu_2(r^2-1)+2}N/(\log N)^{1.005}$. $\square$

Lemma 6. *Let $r \geq 3$ be odd. For every integer $d \geq 1$ and integer $N \geq 1$,*

$$\#\{1 \leq g \leq N : \nu_2(r^g - 1) = d\} \ \leq \ 2^{\,\nu_2(r^2-1)-1}\, \frac{N}{2^{\,d}}.$$

Proof. The 2-adic valuation of $r^g - 1$ is given by Lifting-the-Exponent lemma,

$$\nu_2(r^g - 1) = \begin{cases} \nu_2(r-1) & \text{if } g \text{ is odd,} \\ \nu_2(r^2-1) + \nu_2\big(g/2\big) & \text{if } g \text{ is even.} \end{cases}$$

Let $G = \{1 \leq g \leq N : \nu_2(r^g - 1) = d\}$. If $d < \nu_2(r^2 - 1)$ and $d \neq \nu_2(r-1)$, there is no g in G, hence $\#G = 0$ and the lemma is proved. $\nu_2(r^g - 1) = d$, Now assume the set G is non-empty. Then either $d = \nu_2(r-1)$ or $d \geq \nu_2(r^2-1)$. If $d \geq \nu_2(r^2-1)$, then g must be even and $\nu_2(g/2) = d - \nu_2(r^2-1)$; hence g is a multiple of $2^{\,d-\nu_2(r^2-1)+1}$. The number of such $g \leq N$ is at most

$$\frac{N}{2^{\,d-\nu_2(r^2-1)+1}} = 2^{\,\nu_2(r^2-1)-1}\, \frac{N}{2^{\,d}}.$$

If $d = \nu_2(r-1)$, then g must be odd, so there are at most N such g. In this case the right-hand side of the claimed inequality equals $N \cdot 2^{\,\nu_2(r+1)-1}$, which is greater than or equal to N. $\square$

Lemma 7. *Let $\mathcal{S}$ be a sparse set. Then, for every odd integer $r \geq 3$, every $N \geq \delta_7$ and every non-zero integer h with $|h| \leq \log\log N$,*

$$\int_{\mathcal{C}(\mathcal{S})} \Big|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i r^j h x}\Big|^2 \,\mathrm{d}\mu(x) \leq \frac{2^{\,2\nu_2(r^2-1)+4}}{(\log N)^{1.005}},$$

where δ_7 is the smallest integer greater than or equa to 3 satisfying $\delta_7 \geq \delta_4\big(\lfloor 3\log_2 \log \delta_7 \rfloor + 1\big)$, with δ_4 from Lemma 4.

Proof. Write $h = 2^{\nu_2(h)}u$ with u odd. Expanding the square and using $|\widehat{\mu}(t)| = \prod_{k\in\mathcal{S}} |\cos(\pi t/2^k)|$ gives

$$\int_{\mathcal{C}(\mathcal{S})} \Big|\sum_{j=1}^{N} e^{2\pi i r^j h x}\Big|^2 \,\mathrm{d}\mu(x) \leq \sum_{p=1}^{N}\sum_{q=1}^{N} \prod_{k\in\mathcal{S}} \big|\cos\big(\pi h(r^p - r^q)/2^k\big)\big|.$$

The diagonal $p = q$ contributes N, since $(r^p - r^q) = 0$ and $\cos(0) = 1$.

For $p > q$ set $g = p - q \geq 1$ and $d(g) = \nu_2(r^g - 1)$. Factoring $h(r^p - r^q) = 2^{\nu_2(h)+d(g)} \cdot \ell_g r^q$ with ℓ_g odd, the product becomes

$$\prod_{\substack{k>\nu_2(h)+d(g)\\ k\in\mathcal{S}}} \left|\cos\left(\pi\ell_g r^{\,q}/2^{\,k-\nu_2(h)-d(g)}\right)\right|.$$

Thus, the off-diagonal terms become

$$2\sum_{g=1}^{N-1}\sum_{q=1}^{N-g}\prod_{\substack{k>\nu_2(h)+d(g)\\ k\in\mathcal{S}}} \left|\cos(\pi\ell_g r^{\,q}/2^{\,k-\nu_2(h)-d(g)})\right|.$$

Set the threshold $B = \lfloor 2\log_2(\log N)\rfloor$.

Small valuations $d(g) \leq B$: For g such that $d(g) \leq B$ we extend the inner sum harmlessly to the full block $q = 0, \ldots, N-1$ and apply Lemma 5 with $a = \nu_2(h) + d$ and odd $\ell = \ell_g$,

$$\begin{aligned}\sum_{q=0}^{N-g}\prod_{\substack{k>\nu_2(h)+d(g)\\ k\in\mathcal{S}}} \left|\cos(\pi\ell_g r^{\,q}/2^{\,k-\nu_2(h)-d(g)})\right| &\leq \sum_{q=0}^{N-1}\prod_{\substack{k>\nu_2(h)+d(g)\\ k\in\mathcal{S}}} \left|\cos(\pi\ell_g r^{\,q}/2^{\,k-\nu_2(h)-d(g)})\right| \\ &\leq 2^{\nu_2(r^2-1)+2}\,\frac{N}{(\log N)^{1.005}},\end{aligned}$$

provided $N \geq \delta_4(\nu_2(h) + d(g))$. Let δ_7 be the smallest integer ≥ 3 satisfying $\delta_7 \geq \delta_4\big(\lfloor 3\log_2\log\delta_7\rfloor + 1\big)$. Since the function $f(N) = \delta_4\big(\lfloor 3\log_2\log N\rfloor + 1\big)$ grows slower than any positive power of N (indeed $\log f(N) = O((\log\log N)^{1/\rho})$), the inequality $N \geq f(N)$ persists for all $N \geq \delta_7$. Hence for every $N \geq \delta_7$ we have $N \geq \delta_4(\nu_2(h) + d(g))$, and Lemma 5 applies. Now we count how many g have a given valuation $d(g) = d$. By Lemma 6,

$$\#\{g \leq N : d(g) = d\} \leq 2^{\nu_2(r^2-1)-1}\,\frac{N}{2^{\,d}}.$$

Summing over $d = 1, \ldots, B$ gives at most $2^{\nu_2(r^2-1)-1}N$ such g. Consequently, the total contribution of all small-valuation terms is at most

$$2^{\nu_2(r^2-1)+2}\frac{N}{(\log N)^{1.005}} \cdot 2^{\nu_2(r^2-1)-1}N = 2^{2\nu_2(r^2-1)+1}\,\frac{N^2}{(\log N)^{1.005}}.$$

Large valuations $d(g) > B$. Here we bound each cosine factor by 1; each inner sum is $\leq N$. The number of g such that $d(g) > B$ is

$$\#\{g \leq N : d(g) > d\} \leq \sum_{d>B} 2^{\nu_2(r^2-1)-1}\frac{N}{2^{\,d}} = 2^{\nu_2(r^2-1)-1}\frac{N}{2^{\,B}}.$$

Since $B = \lfloor 2\log_2(\log N)\rfloor$, we have $2^{\,B} \geq \frac{1}{2}(\log N)^2$ for all $N \geq 3$. Thus, the total contribution of all large valuation terms is at most

$$2^{\nu_2(r^2-1)-1}\frac{N}{2^{\,B}}\,N \leq 2^{\nu_2(r^2-1)}\frac{N^2}{(\log N)^2} \leq 2^{\nu_2(r^2-1)}\frac{N^2}{(\log N)^{1.005}},$$

where the last inequality uses $(\log N)^2 \geq (\log N)^{1.005}$ for $N \geq 3$.

Combining the estimates. The double sum (before the symmetry factor 2) satisfies

$$\sum_{g=1}^{N-1}\sum_{q=1}^{N-g}\cdots \le \left(2^{2\nu_2(r^2-1)+1} + 2^{\nu_2(r^2-1)}\right)\frac{N^2}{(\log N)^{1.005}}.$$

Multiplying by 2 gives the total off-diagonal bound

$$2\sum_{g=1}^{N-1}\sum_{q=1}^{N-g}\cdots \le 2^{2\nu_2(r^2-1)+3}\frac{N^2}{(\log N)^{1.005}}.$$

Finally, adding the diagonal N and dividing by N^2 yields, for $N \ge \delta_7 \ge 3$,

$$\int_{\mathcal{C}(\mathcal{S})}\Big|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i r^j hx}\Big|^2 \,\mathrm{d}\mu(x) \le \frac{1}{N} + \frac{2^{2\nu_2(r^2-1)+3}}{(\log N)^{1.005}} \le \frac{2^{2\nu_2(r^2-1)+4}}{(\log N)^{1.005}}.$$

because for $N \ge 3$ we also have $1/N \le 1/(\log N)^{1.005}$. □

Proof of Theorem 1. Fix a sparse set $\mathcal{S}$ and the measure μ Fix a non-zero integer $h = 2^c u$ with u odd. By Weyl's criterion and the Davenport–Erdős–LeVeque lemma [12], it suffices to show

$$\sum_{N\ge 1}\frac{1}{N}\int_{\mathcal{C}(\mathcal{S})}\Big|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i r^n hx}\Big|^2 \mathrm{d}\mu(x) < \infty.$$

By Lemma 7 for every $N \ge \delta_7$ with $|h| \le \log\log N$,

$$\int_{\mathcal{C}(\mathcal{S})}\Big|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i r^j hx}\Big|^2 \,\mathrm{d}\mu(x) \le \frac{2^{\,2\nu_2(r^2-1)+4}}{(\log N)^{1.005}}.$$

Choose $N_0 = N_0(h)$ satisfying $N_0 \ge \delta_7$, the constant set in Lemma 7, and $|h| \le \log\log N_0$; then the estimate holds for all $N \ge N_0$. Hence,

$$\sum_{N\ge 1}\frac{1}{N}\int_{\mathcal{C}(\mathcal{S})}\Big|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i r^j hx}\Big|^2 \,\mathrm{d}\mu(x) \le \sum_{N=1}^{N_0-1}\frac{1}{N} \;+\; 2^{\,2\nu_2(r^2-1)+4}\sum_{N\ge N_0}\frac{1}{N(\log N)^{1.005}} < \infty,$$

because the second series converges by the integral test. Thus, the whole series converges, and by the Davenport–Erdős–LeVeque lemma,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i r^n hx} = 0, \qquad \text{for } \mu\text{-almost every } x.$$

Since this holds for every non-zero integer h, a countable intersection of full μ-measure sets shows that for μ-almost every $x \in \mathcal{C}(\mathcal{S})$ the sequence $(r^n x)_{n\ge 1}$ is equidistributed modulo 1. □

3 Proof of Theorem 2

The following lemma derives immediately from an example due to Bugeaud [10, p. 935].

Lemma 8. *Let $\mathcal{S}$ be a sparse set, let b be an even integer and let α be a real number such that $0 < \alpha < \log_b 2$. Let $(n_i)_{i\geq 1}$ be a strictly increasing sequence of positive integers in $\mathcal{S}$ such that $n_i > (n_{i-1})/\alpha$ for all sufficiently large i. Let $x = \sum_{i\geq 1} 2^{-n_i}$. Then, for every i large enough, among the first $\lfloor \alpha n_i \rfloor$ digits of the base-b expansion of x, there are at most n_{i-1} non-zero digits.*

Proof. Set $c = \log_b 2$ and write $b = 2m$, where $m = b/2$ is an integer. Note that $0 < \alpha < c < 1$ and $\log_b m = \log_b(b/2) = 1 - c$. For any positive integer n, we have

$$2^{-n} = \frac{m^n}{b^n}.$$

The integer m^n has at most $L(n) = \lfloor n \log_b m \rfloor + 1 = \lfloor n(1-c) \rfloor + 1$ digits in base b,

$$b_{L(n)} \dots b_1.$$

Dividing m^n by b^n shifts m^n base-b expansion n places to the right,

$$0.\overbrace{0\dots0}^{n-L(n)}\, b_{L(n)} \dots\dots b_1.$$

So the base-b expansion of 2^{-n} can have non-zero digits only at positions

$$n - L(n) + 1,\ n - L(n) + 2,\ \dots,\ n.$$

The first of these positions is

$$n - \lfloor n(1-c) \rfloor = \lceil nc \rceil \geq nc.$$

Let i_0 be such that $n_i > n_{i-1}/\alpha$ for all $i \geq i_0$. We prove the statement for all sufficiently large i; in particular assume $i > i_0$. Then $n_{i-1} < \alpha n_i$, and since n_{i-1} is an integer,

$$n_{i-1} \leq \lfloor \alpha n_i \rfloor.$$

Now consider the base-b expansion of $x = \sum_{j\geq 1} 2^{-n_j}$. We examine the contributions to the first $\lfloor \alpha n_i \rfloor$ digits.

Terms with $j < i$: For each such term, the non-zero digits of 2^{-n_j} lie at positions $\leq n_j \leq n_{i-1}$. Since $n_{i-1} \leq \lfloor \alpha n_i \rfloor$, all these digits fall within the first $\lfloor \alpha n_i \rfloor$ positions. The total number of non-zero digits contributed by these terms is at most $\sum_{j=1}^{i-1} L(n_j)$.

Terms with $j \geq i$: The first non-zero digit of 2^{-n_j} is at position $\geq cn_j \geq cn_i$. Since $c > \alpha$, we have $cn_i > \alpha n_i$, so the first digit of any term with $j \geq i$ occurs strictly after position $\lfloor \alpha n_i \rfloor$. Hence these terms contribute no non-zero digits to the first $\lfloor \alpha n_i \rfloor$ positions.

It remains to bound $\sum_{j=1}^{i-1} L(n_j)$. Using $L(n_j) \leq n_j(1-c) + 1$, we obtain

$$\sum_{j=1}^{i-1} L(n_j) \leq (1-c) \sum_{j=1}^{i-1} n_j + (i-1).$$

We now estimate $\sum_{j=1}^{i-1} n_j$. From the growth condition $n_k > n_{k-1}/\alpha$, which is equivalent to $n_{k-1} < \alpha n_k$, we have for all k with $i_0 < k \le i-1$,

$$n_{k-1} < \alpha n_k.$$

Iterating this yields, for $i_0 \le j < i$,

$$n_j < \alpha^{i-1-j} n_{i-1}.$$

Split the sum into two parts:

$$\sum_{j=1}^{i-1} n_j = \sum_{j=1}^{i_0-1} n_j + \sum_{j=i_0}^{i-1} n_j.$$

The first sum is a constant C (independent of i). For the second sum,

$$\sum_{j=i_0}^{i-1} n_j < n_{i-1} \sum_{j=i_0}^{i-1} \alpha^{i-1-j} < n_{i-1} \sum_{k\ge 0} \alpha^k = \frac{n_{i-1}}{1-\alpha}.$$

Thus

$$\sum_{j=1}^{i-1} n_j < C + \frac{n_{i-1}}{1-\alpha}.$$

Substituting back,

$$\sum_{j=1}^{i-1} L(n_j) < (1-c)\left(C + \frac{n_{i-1}}{1-\alpha}\right) + (i-1).$$

Set $\delta = \frac{c-\alpha}{1-\alpha} > 0$. Then

$$(1-c)\frac{n_{i-1}}{1-\alpha} = \left(1 - \frac{c-\alpha}{1-\alpha}\right) n_{i-1} = n_{i-1} - \delta n_{i-1}.$$

Hence

$$\sum_{j=1}^{i-1} L(n_j) < n_{i-1} - \delta n_{i-1} + (1-c)C + (i-1).$$

Since $\alpha < 1$, the condition $n_i > n_{i-1}/\alpha$ forces the sequence (n_i) to grow at least geometrically, so $n_{i-1} \to \infty$ exponentially fast as $i \to \infty$. Consequently, the term $-\delta n_{i-1}$ dominates $(1-c)C + (i-1)$, and for all sufficiently large i we have

$$\delta n_{i-1} \ge (1-c)C + (i-1),$$

which implies

$$\sum_{j=1}^{i-1} L(n_j) < n_{i-1}.$$

Since the left-hand side is an integer, we conclude

$$\sum_{j=1}^{i-1} L(n_j) \leq n_{i-1}.$$

Therefore, for all sufficiently large i, the first $\lfloor \alpha n_i \rfloor$ digits of x in base b contain at most n_{i-1} non-zero digits. □

Proof of Theorem 2. By Lemma 8 if we choose the sequence (n_i) so that $\frac{n_i - 1}{n_i} \to 0$ (possible because we may take it to grow arbitrarily fast), then the bound on non-zero digits forces the frequency of zeros in the first$\lfloor \alpha n_i \rfloor$ digits to tend to 1. Then $x = \sum_{i\geq 1} 2^{-n_i}$ is not normal in base b nor in any other even base $b' < b$ because $\log_b 2 < \log_{b'} 2$.

Finally, since $\mathcal{S}$ is sparse, it contains uncountably many strictly increasing sequences (m_k) with $m_{k+1} > (m_k)/\alpha$ and $m_{k-1}/m_k \to 0$. For any infinite such subsequence (m_{k_i}) define $x = \sum_{i\geq 1} 2^{-m_{k_i}} \in \mathcal{C}(\mathcal{S})$. Since different subsequences give different numbers, we conclude that $\mathcal{C}(\mathcal{S})$ contains uncountably many numbers that are not normal in every even base $\leq b$. □

4 Proof of Theorem 3

The measure μ on the unit interval $[0, 1]$ supported on

$$\mathcal{C}(\mathcal{S}) = \left\{ \sum_{n\in S} b_n 2^{-n} : b_n \in \{0, 1\} \right\}.$$

For a dyadic interval $I = (a2^{-n}, (a+1)2^{-n})$ such that the binary expansion of $a2^{-n} = 0.a_1 \cdots a_n$,

$$\mu(I) = \begin{cases} 2^{-S(1,n)}, & \text{if for all } i \text{ such that } 1 \leq i \leq n \text{ , } a_i = 1 \text{ implies } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Let $I = (0.a_1...a_n, 0.a_1...a_n + 2^{-n})$ be a positive-μ-measure dyadic interval in $\mathcal{C}(\mathcal{S})$ and let $I^0 = (0.a_1...a_n0, 0.a_1...a_n1)$ and $I^1 = (0.a_1...a_n1, 0.a_1...a_n1 + 2^{-(n+1)})$, the left and right halves of I respectively. Then,

$$\mu(I^0) = \begin{cases} \mu(I), & n+1 \notin \mathcal{S}, \\ \mu(I)/2, & n+1 \in \mathcal{S}, \end{cases}$$

$$\mu(I^1) = \begin{cases} 0, & n+1 \notin \mathcal{S}, \\ \mu(I)/2, & n+1 \in \mathcal{S}. \end{cases}$$

Therefore, both halves have positive measure exactly when $n + 1 \in \mathcal{S}$. If $n + 1 \notin \mathcal{S}$, the half ending in 1 has measure zero.

Any open set in the Cantor-type set $\mathcal{C}(\mathcal{S})$ can be written as a countable disjoint union of sets of the form $\mathcal{C}(\mathcal{S}) \cap I$, where I is a dyadic interval. Since μ is countably additive, if $X \subseteq \mathcal{C}(\mathcal{S})$ and $\mu(X) > 0$ then at least one of those dyadic intervals must have positive μ-measure. Endpoints have μ-measure zero because they are intersections of nested dyadic intervals with measures tending to 0; hence the result holds for all interval types.

For every integer $m \geq 1$ we let N_n denote the number of terms to be considered in the exponential sum and set

$$N_m = \lfloor 2^{m^\alpha} \rfloor,$$

where $\alpha = 0.999$ (actually any fixed α such that $1/1.005 < \alpha < 1$ works). So, N_m is subexponential in m, $\log N_m = m^\alpha \log 2 + O(1)$, and $\log\log N_m = \log(m^\alpha \log 2 + O(1)) \leq 2 \log m$ for sufficiently large m.

Lemma 9. *For every odd $r \geq 3$, for every $m \geq \delta_9(r)$ and every non-zero integer h with $|h| \leq \log\log N_m$,*

$$\int_{\mathcal{C}(\mathcal{S})} \Big| \frac{1}{N_m} \sum_{j=1}^{N_m} e^{2\pi i r^j h x} \Big|^2 \, \mathrm{d}\mu(x) \leq \frac{C_r}{m^{1.003}}.$$

where $\delta_9(r) = \max\{\big(\log_2 \delta_7(r)\big)^{1/\alpha}, 3\}$ and $C_r = 2^{\,2\nu_2(r^2-1)+4}/(\frac{1}{2}\log 2)^{1.005}$.

Proof. Let $\delta_9(r) = \max\{\big(\log_2 \delta_7(r)\big)^{1/\alpha}, 3\}$. Recall $N_m = \lfloor 2^{m^\alpha} \rfloor$ with $\alpha = 0.999$. Since $m \geq \delta_9(r)$ then $N_m \geq \delta_7(r)$. By Lemma 7 with $N = N_m$,

$$\int_{\mathcal{C}(\mathcal{S})} \Big| \frac{1}{N_m} \sum_{j=1}^{N_m} e^{2\pi i r^j h x} \Big|^2 \, \mathrm{d}\mu(x) \leq \frac{2^{\,2\nu_2(r^2-1)+4}}{(\log N_m)^{1.005}}.$$

Since $N_m \geq 2^{m^\alpha} - 1$ and for $m \geq 3$ we have $2^{m^\alpha} \geq 2^{3^{0.999}} > 8$, it follows that for all $m \geq 3$,

$$\log N_m \geq m^\alpha \log 2 - \log 2 \geq \frac{1}{2} m^\alpha \log 2.$$

Therefore, for $m \geq \delta_9(r) \geq 3$,

$$\frac{2^{\,2\nu_2(r^2-1)+4}}{(\log N_m)^{1.005}} \leq \frac{2^{\,2\nu_2(r^2-1)+4}}{(\frac{1}{2}\log 2)^{1.005}} \frac{1}{m^{1.005\alpha}} \leq \frac{C_r}{m^{1.003}},$$

where $C_r = 2^{\,2\nu_2(r^2-1)+4}/(\frac{1}{2}\log 2)^{1.005}$ and we used $1.005\alpha = 1.003995 > 1.003$ to obtain for all $m \geq 1$, $1/m^{1.005\alpha} \leq 1/m^{1.003}$. $\square$

Define for each non-zero-integer h, odd $r \geq 3$ and integer $m \geq 1$,

$$f_{h,r,m}(x) = \frac{1}{N_m} \sum_{j=1}^{N_m} e^{2\pi i r^j h x},$$

$$\Lambda_{r,m} = \bigcup_{|h|=1}^{\log\log N_m} \Big\{ x \in \mathcal{C}(\mathcal{S}) : \Big| f_{h,r,m}(x) \Big| > \frac{1}{\log\log N_m} \Big\}.$$

Remark 1. Each $\Lambda_{r,m}$ is a finite union of intervals. For each h, r, m, the boundary condition $|f_{h,r,m}(x)| = 1/\log\log N_m$ is equivalent to a polynomial equation in $\cos(2\pi x)$ with rational coefficients; hence the endpoints are algebraic numbers and computable. so the endpoints are computable; consequently, $\mu(\Lambda_{r,m})$ is computable uniformly in r, m.

Lemma 10. *For all $m \geq \delta_9(r)$,*

$$\mu(\Lambda_{r,m}) \leq C'_r \frac{(\log m)^3}{m^{1.003}}.$$

where $C'_r = 16C_r$ with C_r and $\delta_9(r)$ determined in Lemma 9.

Proof. By the generalized Markov/Chebyshev inequality,

$$\mu(\{x \in X \,:\, |f(x)| > T\}) \quad < \frac{1}{T^2} \int_X |f(x)|^2 \, d\mu(x).$$

Let $T = 1/(\log\log N_m)$. By Lemma 9, for all $m \geq \delta_9(r) \geq 3$ and every non-zero integer h with $|h| \leq \log\log N_m$,

$$\int_{\mathcal{C}(\mathcal{S})} |f_{h,r,m}(x)|^2 \, \mathrm{d}\mu(x) \leq C_r/m^{1.003}.$$

Since the set $\Lambda_{r,m}$ is the union over at most $2\log\log N_m$ integers h with $1 \leq |h| \leq \log\log N_m$,

$$\begin{aligned}
\mu(\Lambda_{r,m}) &\leq \frac{1}{(\log\log N_m)^2} \sum_{|h|=1}^{\log\log N_m} \int_{\mathcal{C}(\mathcal{S})} |f_{h,r,m}(x)|^2 \, \mathrm{d}\mu(x) \\
&\leq (\log\log N_m)^2 \cdot 2\log\log N_m \cdot \frac{C_r}{m^{1.003}} = 2C_r \, \frac{(\log\log N_m)^3}{m^{1.003}} \\
&\leq 2C_r \cdot 8 \, \frac{(\log m)^3}{m^{1.003}} = 16\, C_r \, \frac{(\log m)^3}{m^{1.003}}.
\end{aligned}$$

The last inequality follows for $m \geq 3$ because $N_m \leq 2^{m^\alpha}$ with $\alpha < 1$, so $\log N_m \leq m\log 2$, hence $\log\log N_m \leq 2\log m$. $\square$

Remark 2. The series $\sum_{m\geq 1}(\log m)^3/m^{1.003}$ converges (its terms are $O(1/m^{1.003})$). Its tail can be bounded effectively, for instance by comparing with the integral $\int(\log x)^3/x^{1.003}\, dx$, which is a computable decreasing function of m that can be approximated from above by rationals to any desired precision.

Let ε a rational , $0 < \varepsilon < 2/3$, which is an input of Algorithm 1 and remains fixed. For each base odd base r we define $m_0(r)$ as the least integer such that

$$\sum_{m\geq m_0(r)} C'_r \frac{(\log m)^3}{m^{1.003}} < \frac{\varepsilon}{2^r}.$$

where C'_r is determined in Lemma 10. We define

$$\begin{aligned}
\Delta &= \bigcup_{\substack{r\geq 3\\ r \text{ odd}}} \bigcup_{m\geq m_0(r)} \Lambda_{r,m}, \\
s &= \sum_{\substack{r\geq 3\\ r \text{ odd}}} \sum_{m\geq m_0(r)} \mu(\Lambda_{r,m}).
\end{aligned}$$

Then,

$$\mu(\Delta) \le s < \sum_{\substack{r\ge 3\\ r \text{ odd}}} \frac{\varepsilon}{2^r} = \frac{\varepsilon}{6} < \varepsilon.$$

For an integer $t \ge 3$, define the approximating set Δ_t, the partial sum s_t and the tail u_t,

$$\begin{aligned}
\Delta_t &= \bigcup_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} \bigcup_{m=m_0(r)}^{t} \Lambda_{r,m},\\
s_t &= \sum_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} \sum_{m=m_0(r)}^{t} \mu(\Lambda_{r,m}),\\
u_t &= s - s_t.
\end{aligned}$$

Clearly, for every $t \ge 3$, $\mu(\Delta_t) \le s_t$ and $u_t > 0$ with $\lim_{t\to\infty} u_t = 0$.

Remark 3. Let's see that u_t is computable. Since the computable numbers form a field, any finite sum of computable numbers is computable. Each $\mu(\Lambda_{r,m})$ is computable, so for each t, s_t is computable. The sequence $(s_t)_{t\ge 3}$ is nondecreasing and bounded above by s, and $s = \lim_{t\to\infty} s_t = \sup_t s_t$. A standard theorem in computable analysis states that the supremum of a computable, increasing, bounded sequence of computable reals is computable exactly when there exists a computable function $\ell \mapsto t_\ell$ such that $s - s_{t_\ell} < 2^{-\ell}$ for all ℓ. For each $t \ge 3$,

$$u_t = s - s_t = \Big(\sum_{\substack{r>\lfloor \log t\rfloor\\ r \text{ odd}}} \sum_{m\ge m_0(r)} \mu(\Lambda_{r,m})\Big) + \Big(\sum_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} \sum_{m>\max(m_0(r),t)} \mu(\Lambda_{r,m})\Big).$$

The first double sum is bounded by the tail of the convergent geometric series. The exponential decay in r converts the logarithmic threshold $r > \log t$ into $1/t$ decay,

$$\sum_{\substack{r>\lfloor \log t\rfloor\\ r \text{ odd}}} \frac{\varepsilon}{2^r} = \frac{2\varepsilon}{3} \cdot 2^{-\lfloor \log t\rfloor} < (4\varepsilon/3t^{\log 2}).$$

The second double sum decays to zero polynomially with a very small exponent,

$$\begin{aligned}
\sum_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} \sum_{m>\max\{m_0(r),t\}} \mu(\Lambda_{r,m}) &\le \sum_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} C'_r \sum_{m>\max\{m_0(r),t\}} \frac{(\log m)^3}{m^{1.003}}\\
&\le K \frac{(\log t)^3}{t^{0.003}} \sum_{\substack{r=3\\ r \text{ odd}}}^{\lfloor \log t\rfloor} C'_r\\
&\le K' \frac{(\log t)^8}{t^{0.003}}
\end{aligned}$$

where $C'_r = 162^{\,2\nu_2(r^2-1)+4}/(\frac{1}{2}\log 2)^{1.005}$ and K' is a positive constant. Hence we can effectively choose t_ℓ such that $u_{t_\ell} < 2^{-\ell}$, providing the required computable modulus of convergence. Therefore s is computable. Finally, for each fixed t, we have $u_t = s - s_t$. Since the computable numbers are a field and both s and s_t are computable reals, their difference u_t is computable.

Lemma 11. *For any interval I and any integers t, ℓ with $3 \le t \le \ell$,*

1. $\mu(\Delta \setminus \Delta_t) \le u_t$.
2. $\mu(\Delta_\ell \setminus \Delta_t) \le u_t - u_\ell$.
3. $\mu(\Delta \cap I) \le \mu(\Delta_t \cap I) + u_t$.
4. $\mu(\Delta_\ell \cap I) \le \mu(\Delta_t \cap I) + u_t - u_\ell$.

Proof. These follow directly from the definitions of Δ, Δ_t, s_t, u_t and subadditivity. □

We now give the algorithm that proves Theorem 3.

Algorithm 1. Let $k_1 < k_2 < k_3 < \cdots$ be the elements of $\mathcal{S}$ in increasing order. Given positive rational $\varepsilon < 2/3$, the following algorithm defines the binary expansion $x = x(\varepsilon) = 0.b_1b_2b_3\dots$ digit by digit.

Initialisation $i = 0$. For $j = 1, \dots, k_1 - 1$ set $b_j = 0$. Set $I_0 = [0, 2^{-(k_1-1)})$, so $|I_0| = 2^{-(k_1-1)}$ and $\mu(I_0) = 1$. Choose p_0 large enough so that $u_{p_0} < \varepsilon$.

Step $i \ge 1$. For $j = k_{i-1} + 1, .., k_i - 1$ set $b_j = 0$. The current interval is I_{i-1}, with $|I_{i-1}| = 2^{-(k_i-1)}$ and $\mu(I_{i-1}) = 2^{-(i-1)}$.

Split I_{i-1} into its left half I^0_{i-1} and its right half I^1_{i-1}. Since $k_i \in \mathcal{S}$, $\mu(I^0_{i-1}) = \mu(I^1_{i-1}) = 2^{-i}$.

Choose $p_i \ge p_{i-1}$ such that $u_{p_i} < 2^{-2i}\varepsilon$.

Set

$$b_{k_i} = \begin{cases} 0 & \text{if } \mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + u_{p_i} < 2^{-i}(\varepsilon + \sum_{j=1}^{i} 2^{j-1} u_{p_j}), \text{is verified first,} \\ 1 & \text{if } \mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) + u_{p_i} < 2^{-i}(\varepsilon + \sum_{j=1}^{i} 2^{j-1} u_{p_j}), \text{is verified first.} \end{cases}$$

Set $I_i = I^{b_{k_i}}_{i-1}$.

The following lemma proves the invariant of the algorithm: at every step i, the interval I_i contains a positive μ-measure of numbers that are not in Δ.

Lemma 12. *For every $i \ge 0$, $\mu(I_i \setminus \Delta) > 0$.*

Proof. We prove the lemma by induction on the number of steps i. Let

$$T_i = \varepsilon + \sum_{j=1}^{i} 2^{j-1} u_{p_j}.$$

Induction hypothesis (IH): For every $i \ge 0$,

$$\mu(\Delta_{p_i} \cap I_i) + u_{p_i} < \frac{1}{2^i} T_i.$$

Base case $i = 0$. Since $\mu(\Delta_{p_0}) \le s_{p_0}$, by subadditivity, Hence

$$\mu(\Delta_{p_0} \cap I_0) + u_{p_0} \le s_{p_0} + u_{p_0} = s < \varepsilon = T_0.$$

Inductive step. Assume (IH) for $i-1$. Since $I^0_{i-1} \cup I^1_{i-1} = I_{i-1}$,

$$\mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + \mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) = \mu(\Delta_{p_i} \cap I_{i-1})\,.$$

Apply Lemma 11 point (4) with $t = p_{i-1}$, $\ell = p_i$, $I = I_{i-1}$:

$$\mu(\Delta_{p_i} \cap I_{i-1}) \leq \mu\big(\Delta_{p_{i-1}} \cap I_{i-1}\big) + u_{p_{i-1}} - u_{p_i}.$$

Adding $2u_{p_i}$,

$$(\mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + u_{p_i}) + (\mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) + u_{p_i}) \leq \mu\big(\Delta_{p_{i-1}} \cap I_{i-1}\big) + u_{p_{i-1}} + u_{p_i}.$$

By (IH) for $i-1$,

$$\mu\big(\Delta_{p_{i-1}} \cap I_{i-1}\big) + u_{p_{i-1}} < \frac{1}{2^{\,i-1}} T_{i-1}.$$

Hence

$$(\mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + u_{p_i}) + (\mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) + u_{p_i}) < \frac{1}{2^{\,i-1}} T_{i-1} + u_{p_i}.$$

Hence, if both, $\big(\mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + u_{p_i}\big)$ and $\big(\mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) + u_{p_i}\big)$ were greater than or equal to

$$\frac{1}{2}\Big(\frac{1}{2^{\,i-1}} T_{i-1} + u_{p_i}\Big) = \frac{1}{2^i} T_i$$

we would have

$$(\mu\big(\Delta_{p_i} \cap I^0_{i-1}\big) + u_{p_i}) + (\mu\big(\Delta_{p_i} \cap I^1_{i-1}\big) + u_{p_i}) \geq 2\frac{1}{2^i} T_i = \frac{1}{2^{\,i-1}} T_{i-1} + u_{p_i}$$

contradicting the inequality above. Therefore, at least one of them is less than $\frac{1}{2^i} T_i$. The interval $I_i = I^{b_{k_i}}_{i-1}$ chosen by the algorithm satisfies

$$\mu(\Delta_{p_i} \cap I_i) + u_{p_i} < \frac{1}{2^i} T_i.$$

Since $u_{p_i} < 2^{-2i}\varepsilon$,

$$\sum_{j\geq 1} 2^{\,j-1} u_{p_j} < \varepsilon \sum_{j \geq 1} 2^{-j-1} = \frac{\varepsilon}{2}.$$

Thus, $T_i \leq \varepsilon + \varepsilon/2 = (3/2)\varepsilon$ for all i. Since we assumed $\varepsilon < 2/3$, from (IH) and Lemma 11 point (3),

$$\mu(\Delta \cap I_i) \leq \mu(\Delta_{p_i} \cap I_i) + u_{p_i} < \frac{\varepsilon\ 3/2}{2^i} < 2^{-i} = \mu(I_i).$$

We conclude that no I_i is completely covered by Δ. $\square$

Lemma 13. *Let positive rational $\varepsilon < 2/3$. The number $x = x(\varepsilon)$ defined by the algorithm is normal to all odd bases.*

Proof. The nested dyadic intervals I_i shrink to a unique point $x \in \mathcal{C}(\mathcal{S})$, that is, $x = \bigcap_{i \geq 1} I_i$. By Lemma 12, for every $i \geq 0$, $\mu(I_i \setminus \Delta) > 0$, we conclude that $x \notin \Delta$. Take any odd $r \geq 3$ and any integer $h \neq 0$. For all sufficiently large m, we have $|h| \leq \log\log N_m$ and $x \notin \Lambda_{r,m}$, so

$$\Big|\frac{1}{N_m}\sum_{j=1}^{N_m} e^{2\pi i r^{\,j} hx}\Big| \leq \frac{1}{\log\log N_m} \to 0 \quad \text{as } m \to \infty.$$

Now let M be an arbitrary positive integer. Choose the unique m such that

$$N_{m-1} < M \leq N_m,$$

where we set $N_0 = 1$. Then we can write

$$\frac{1}{M}\sum_{j=1}^{M} e^{2\pi i r^{\,j} hx} = \frac{N_m}{M} \cdot \frac{1}{N_m}\sum_{j=1}^{N_m} e^{2\pi i r^{\,j} hx} \;-\; \frac{1}{M}\sum_{j=M+1}^{N_m} e^{2\pi i r^{\,j} hx}.$$

The absolute value of the second summand is at most

$$(N_m - M)/M.$$

Since $N_m = \lfloor 2^{m^\alpha} \rfloor$ is subexponential in m, we have $N_m/N_{m-1} \to 1$. Since $M > N_{m-1}$, we have $N_m/M \leq N_m/N_{m-1} \to 1$, hence $\frac{N_m - M}{M} \to 0$. Thus, the second summand tends to 0. Now we argue that the first summand also tends to 0. Given that $x \notin \Lambda_{r,m}$, we have $\left|\frac{1}{N_m}\sum_{j=1}^{N_m} e^{2\pi i r^{\,j} hx}\right| \leq 1/(\log\log N_m)$ which tends to 0 and $m \to \infty$. Since $N_m/M \to 1$, we conclude

$$\lim_{M\to\infty} \frac{1}{M}\sum_{j=1}^{M} e^{2\pi i r^{\,j} hx} = 0.$$

This is Weyl's criterion for the sequence $(r^{\,j}x)_{j\geq 1}$ to be uniformly distributed modulo 1. As a result, x is normal in base r. □

Lemma 14. *Assume $\mathcal{S}$ is a computable sparse set with a computable sparsity exponent ρ. Assume ε is a positive real number such that $\varepsilon < 2/3$. Then, then the number $x = x(\varepsilon)$ defined by Algorithm 1 is computable.*

Proof. The sparse set $\mathcal{S}$ is computable, so there is a a computable enumeration of all its elements in increasing order. It also implies that the measure μ on dyadic intervals is computable. By Remark 1, for each odd $r \geq 3$ and for each m, $\mu(\Lambda_{r,m})$ is computable. Since the value $m_0(r)$ we have, for each t, $\mu(\Delta_t)$ is is computable. By Remark 3, u_t is computable. Given that $u_t \to 0$, the algorithm can perform a finite search to find the first p_i such that $u_{p_i} < 2^{-2i}\varepsilon$. Since Δ_{p_i} is a finite union of intervals with computable endpoints, so is $\Delta_{p_i} \cap I^0_{i-1}$. The measure μ on this set is computable. The choice of b_{k_i} is the result of a comparison of computable reals. The decision whether a given real number is strictly less than another is computable unless they are equal. By the proof of Lemma 12, at least one of I^0_{i-1} or I^1_{i-1} satisfies the strict inequality condition. □

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