# Normal numbers in sparse Cantor sets

Verónica Becher
University of Buenos Aires

joint work with Simón Lew Deveali

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# Expansion of a real number in an integer base

For a real number $x$, its expansion in an integer base $b \geq 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leq a_n < b$ for every $n$, such that

$$x - \lfloor x \rfloor = \sum_{n \geq 1} a_n b^{-n} = 0.a_1 a_2 a_3 \ldots$$

We require that $a_n < b - 1$ infinitely often to ensure that every number has a unique representation.

# Borel normal numbers

A base is an integer $b \geq 2$.

A real number $x$ is normal to base $b$ if, in the base-$b$ expansion of $x$, every block of digits of the same length occurs with the same asymptotic frequency (that is, with frequency $1/b^{\text{block length}}$).

Borel proved that almost all numbers, with respect to Lebesgue measure, are normal to all integer bases.

# Examples of Borel normal numbers

$0.01010101010\ldots$ is simply normal to base $2$ but not normal to base $2$.

Each number in middle third Cantor set is not simply normal to base $3$.

Champernowne's $0.12345679101112131415161718192021\ldots$
is normal to base $10$, but is not known whether it is normal to bases multiplicatively independent to $10$.

Stoneham number $\alpha_{2,3} = \sum_{n\geq 1} \frac{1}{3^n \; 2^{3^n}}$ is normal to base $2$ but not simply normal to base $6$ (Bailey and Borwein, 2012).

There are computable constructions of (absolutely) normal numbers.

# Normal numbers in sparse Cantor sets

## Definition (Sparse set)

*A set $\mathcal{S}$ of positive integers is sparse if it has density zero and there exists a sparsity exponent $\rho > 0$, such that*

$$\liminf_{k\to\infty} \min_{0\le a\le k^\rho} \frac{\#(\mathcal{S}\cap[a,a+k]}{30\log k} > 1,$$

Examples: The prime numbers, the triangular numbers, the squares.
The numbers in $\{\lceil e^{j/100}\rceil : j \ge 1\}$.
Neither the powers of two, nor the factorials are sparse sets.

## Definition (Sparse Cantor set $\mathcal{C}(\mathcal{S})$)

$$\mathcal{C}(\mathcal{S}) = \Big\{x \in [0,1) : x = \sum_{k\in\mathcal{S}} b_k 2^{-k}\Big\}, b_k \in \{0,1\}\}$$

**Theorem 1** (Becher and Lew Deveali 2026)

*Let $\mathcal{S}$ be a sparse set of positive integers. There are uncountably many numbers in $\mathcal{C}(\mathcal{S})$ that are normal in every odd base.*

**Theorem 2** (Becher and Lew Deveali 2026)

*Let $\mathcal{S}$ be a sparse set of positive integers. For any even base $b$, there are uncountably many numbers in $\mathcal{C}(\mathcal{S})$ that are not normal in any even base $\leq b$.*

**Theorem 3** (Becher and Lew Deveali 2026)

*Let $\mathcal{S}$ be a computable sparse set of positive integers with a computable sparsity exponent. There is an algorithm that given a rational $\varepsilon$, $0 < \varepsilon < 2/3$, produces a number $x(\varepsilon)$ in $\mathcal{C}(\mathcal{S})$ expressed in base two, that is normal in all odd bases.*

# About Theorem 1

Cassels (1959): there are uncountably many numbers in the ternary Cantor set that are normal to all bases mutiplicatively independent to $3$.

Schmidt (1961/1962): for any set of bases $R$ closed by mutiplicative dependence, an explicit construction of uncountably many numbers that are normal every base $r \in R$ and not normal to every base $s \notin R$.

Volkmann (1984): for multiplicatively independent bases $r$ and $r$, the existence of uncountably many numbers normal to base $r$ while having given asymptoptic frequency of digits in base $s$, In particular, digit has asymptotic frequency 1 Uses a specific Cantor type set made of 3 digits, constructed from the triangular numbers.

In contrast, Theorem 1 is uniform on every sparse cantor set $\mathcal{C}(\mathcal{S})$ for odd $r$.

# Proof of Theorem 1

## Definition (Measure $\mu$ on $\mathcal{C}(\mathcal{S})$)

*Given a sparse set $\mathcal{S}$ of positive integers, we define a probability measure $\mu = \mu(\mathcal{S})$ on the unit interval $[0,1)$ equipped with the Borel $\sigma$-algebra, via the binary expansion of numbers. The numbers $x \in \mathcal{C}(\mathcal{S})$ are of the form $x = \sum_{k \geq 1} b_k 2^{-k}$, where, for every $k \geq 1$, digit $b_k = 0$ with probability $1/2$ if $k \in \mathcal{S}$, and digit $b_k = 0$ with probability $1$ otherwise.*

We redevelop Cassels-Schmidt proofs, to hold uniformly on every $\mathcal{C}(\mathcal{S})$, for sparse $S$.

# Proof of Theorem 1

To show that $\mu$-almost all numbers in $\mathcal{C}(\mathcal{S})$ are normal to odd base $r$ we apply a lemma of Davenport, Erdös and LeVeque. It says :

If, for each integer $h \neq 0$, the series

$$\sum_{N \geq 1} \frac{1}{N} \int_{\mathcal{C}(\mathcal{S})} \left| \frac{1}{N} \sum_{n=}^{N-1} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) \text{ converges}$$

then $\mu$-almost all elements $x$ of $\mathcal{C}(\mathcal{S})$ satisfy $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{0 \leq n < N} e^{2\pi i r^n h x} = 0$; hence, they are normal to base $r$.

Fix $h \neq 0$. Let $X = \sum_{k \in \mathcal{S}} b_k 2^{-k}$ the random variable with distribution $\mu$, so the $b_k$ are independent and $\mathbb{P}(b_k = 0) = p_k$, $\mathbb{P}(b_k = 1) = 1 - p_k$, where $p_k = 1/2$ if $k \in \mathcal{S}$ and $p_k = 1$, otherwise.

$$\int_{\mathbb{C}} \left| \sum_{0 \leq n < N} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) \qquad = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h (r^m - r^n) x} \, d\mu(x) =$$

Fix $h \neq 0$. Let $X = \sum_{k \in \mathcal{S}} b_k 2^{-k}$ the random variable with distribution $\mu$, so the $b_k$ are independent and $\mathbb{P}(b_k = 0) = p_k$, $\mathbb{P}(b_k = 1) = 1 - p_k$, where $p_k = 1/2$ if $k \in \mathcal{S}$ and $p_k = 1$, otherwise.

$$\int_{\mathbb{C}} \left| \sum_{0 \leq n < N} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h (r^m - r^n) x} \, d\mu(x) =$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\Big[ e^{-2\pi i (r^m - r^n) h X} \Big] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\Big[ e^{-2\pi i (r^m - r^n) h \sum_{k \in \mathcal{S}} b_k 2^{-k}} \Big] =$$

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$$\int_{\mathbb{C}} \left| \sum_{0\leq n<N} e^{2\pi i r^n hx} \right|^2 \mathrm{d}\mu(x) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h(r^m - r^n)x}\, d\mu(x) =$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\Big[e^{-2\pi i(r^m - r^n)hX}\Big] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\Big[e^{-2\pi i(r^m - r^n)h\sum_{k\in\mathcal{S}} b_k 2^{-k}}\Big] =$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\left[\prod_{k\in\mathcal{S}} e^{-2\pi i(r^m - r^n)hb_k 2^{-k}}\right] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k\in\mathcal{S}} \mathbb{E}\left[\left(e^{-2\pi i(r^m - r^n)h2^{-k}}\right)^{b_k}\right]$$

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$$\int_{\mathbb{C}} \left| \sum_{0 \leq n < N} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h (r^m - r^n) x} \, d\mu(x) =$$

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$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} p_k \left(e^{-2\pi i (r^m - r^n) h 2^{-k}}\right)^0 + (1 - p_k) \left(e^{-2\pi i (r^m - r^n) h 2^{-k}}\right)^1 =$$

Fix $h \neq 0$. Let $X = \sum_{k \in \mathcal{S}} b_k 2^{-k}$ the random variable with distribution $\mu$, so the $b_k$ are independent and $\mathbb{P}(b_k = 0) = p_k$, $\mathbb{P}(b_k = 1) = 1 - p_k$, where $p_k = 1/2$ if $k \in \mathcal{S}$ and $p_k = 1$, otherwise.

$$\int_{\mathbb{C}} \left| \sum_{0 \le n < N} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h (r^m - r^n) x} \, d\mu(x) =$$

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$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} \frac{1}{2} + \frac{1}{2} e^{-2\pi i (r^m - r^n) h 2^{-k}} \quad = \text{by Euler's identity } \cos\theta = (e^{i\theta} + e^{-i\theta})/2$$

Fix $h \neq 0$. Let $X = \sum_{k \in \mathcal{S}} b_k 2^{-k}$ the random variable with distribution $\mu$, so the $b_k$ are independent and $\mathbb{P}(b_k = 0) = p_k$, $\mathbb{P}(b_k = 1) = 1 - p_k$, where $p_k = 1/2$ if $k \in \mathcal{S}$ and $p_k = 1$, otherwise.

$$\int_{\mathbb{C}} \left| \sum_{0 \leq n < N} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^1 e^{-2\pi i h (r^m - r^n) x} \, d\mu(x) =$$

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Fix $h \neq 0$. Let $X = \sum_{k \in \mathcal{S}} b_k 2^{-k}$ the random variable with distribution $\mu$, so the $b_k$ are independent and $\mathbb{P}(b_k = 0) = p_k$, $\mathbb{P}(b_k = 1) = 1 - p_k$, where $p_k = 1/2$ if $k \in \mathcal{S}$ and $p_k = 1$, otherwise.

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$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} p_k \left( e^{-2\pi i (r^m - r^n) h 2^{-k}} \right)^0 \quad + (1 - p_k) \left( e^{-2\pi i (r^m - r^n) h 2^{-k}} \right)^1 =$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} \frac{1}{2} + \frac{1}{2} e^{-2\pi i (r^m - r^n) h 2^{-k}} \qquad = \text{by Euler's identity } \cos\theta = (e^{i\theta} + e^{-i\theta})/2$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} (e^{-2\pi i (r^m - r^n) h 2^{-k}}) \cos(\pi (r^m - r^n) h / 2^k)$$

$$\leq \sum_{m=0}^{N-1} \underbrace{\sum_{n=0}^{N-1} \prod_{k \in \mathcal{S}} |\cos(\pi (r^m - r^n) h / 2^k)|} \qquad \text{by next Key Lemma } \frac{C_r N}{(\log N)^{1.005}}$$

Then, the Davenport Erdös LeVesque expression

$$\sum_{N\geq 1} \frac{1}{N} \int_{\mathcal{C}(\mathcal{S})} \left| \frac{1}{N} \sum_{n=}^{N-1} e^{2\pi i r^n h x} \right|^2 \mathrm{d}\mu(x) \leq \sum_{N\geq 1} \frac{1}{N} \frac{1}{N^2} \left( N + 2 \sum_{p=1}^{N} \frac{C_r N}{(\log N)^{1.005}} \right)$$

converges. The same holds for every integer $h \neq 0$ (the constant $C_r$ is independent of $h$). Hence, $\mu$-almost all $x \in \mathcal{C}(\mathcal{S})$ are normal to base $r$. $\square$

## Key Lemma

*Let $\mathcal{S}$ be a sparse set, let $a \geq 0$ be an integer and let $r \geq 3$ and $\ell > 0$ be odd integers. For every $N \geq \delta_4(a)$,*

$$\sum_{n=0}^{N-1} \prod_{\substack{k>a \\ k\in\mathcal{S}}} \left|\cos(\pi\ell r^{\,n}/2^{\,k-a})\right| \leq C_r \frac{N}{(\log N)^{1.005}}.$$

# Proof of Lemma (redevelop Schmidt's)

For fixed $r, \ell, n$, if $(\ell r^n) = \sum_{i=1}^{q} c_i 2^{i-1} = c_q c_{q-1} \ldots c_2 c_1$ with $c_i \in \{0, 1\}$,

$$
\begin{array}{lll}
\ell r^n / 2^0 = & c_q \cdots\cdots c_1 & \\
\ell r^n / 2^1 = & \quad c_q \cdots\cdots c_2 & .c_1 \\
\ell r^n / 2^2 = & \quad\; c_q \cdots\cdot c_3 & .c_2 c_1 \\
\ell r^n / 2^3 = & \quad\quad c_q \cdots c_4 & .c_3 c_2 c_1 \\
\ldots & &
\end{array}
$$

If for many $k \in \mathcal{S}$ we have $1/4 \leq \{\ell r^n / 2^{k-a}\} \leq 1/2$ then
$\prod_{\substack{k>a \\ k \in \mathcal{S}}} \left| \cos(\pi \ell r^n / 2^{k-a}) \right| \leq (\sqrt{2}/2)^{\text{many}}$ (small).

Question: How do we know $1/4 < \{\alpha\} < 1/2$?
Answer: When $(\{\alpha\})_2 = 0.01\cdots$ or $(\{\alpha\})_2 = 0.10\cdots$

Each integer $x$ such that $0 \leq x < 2^q$ has a binary representation

$$x = \sum_{i=1}^{q} c_i 2^{i-1} = c_q c_{q-1} \dots c_2 c_1 \text{ with } c_i \in \{0, 1\}.$$

The digit pairs of $x$ are the pairs $(c_q, c_{q-1}), (c_{q-1}, c_{q-2}) \dots, (c_2, c_1)$.

For integer $a \geq 0$, $c_{i-a} c_{i-1-a}$ is an $a$-obedient digit if
$i \in \mathcal{S}$ and $c_{i-a} c_{i-1-a} = 01$ or $c_{i-a} c_{i-1-a} = 10$ .

Let $z_a(x)$ denote the number of $a$-obedient digit pairs $c_{i-a}c_{i-1-a}$ of $x$.
For all integers $a \geq 0$ and all integers $q \geq \delta_3(a)$,

$$\#\Big\{0 \leq x < 2^q : z_a(x) < \Big\lfloor 2.9 \log q \Big\rfloor \Big\} < \frac{2^q}{q^2}$$

It follows that

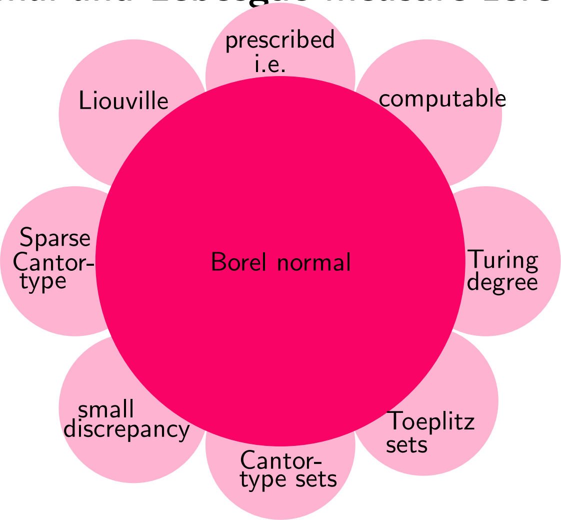
$$\underbrace{\#\{n < N : z_a(\ell r^n) < \lfloor 2.9 \log N \rfloor\}}_{A} \leq C'_r \frac{N}{(\log N)^2}$$

$$\underbrace{\#\{n < N : z_a(\ell r^n) \geq \lfloor 2.9 \log N \rfloor\} \cdot (\sqrt{2}/2))^{\lfloor 2.9 \log N \rfloor}}_{B} \leq C''_r \frac{N}{(\log N)^{1.005}}$$

$$\sum_{n=0}^{N-1} \prod_{\substack{k>a \\ k \in \mathcal{S}}} \left|\cos(\pi \ell r^{\,n}/2^{\,k-a})\right| \leq A + B \leq C_r \frac{N}{(\log N)^{1.005}}$$

□

# Borel normal and Lebesgue measure zero properties




Turing(1937), Cassels (1959), Schmidt (1961/1962), Bugeaud (2002), Levin (1999)

Conjecture (Borel 1951)

All algebraic irrational numbers are absolutely normal.

A number in $[0,1]$ is deterministic in base $b$ if every asymptotic statistical description of its digit sequence is a zero-entropy process—no matter which subsequence you look at, you never see the exponential pattern.

Periodic base-$2$ expansions, numbers with base-$2$ expansion with asymptotic density of $0$s equal to 1, Sturmian numbers, Numbers whose base-$2$ expansion is a paper-folding sequence, ...

Rauzy 1976 proved that a real number is deterministic in base $b$ if and only if adding it to any base-$b$ normal number $x$ preserves normality—that is, $x+y$ remains normal in base $b$.

## Corollary

*There are uncountably many deterministic numbers in base $2$ that are normal in all odd bases; and for any even base $b$, uncountably many that are not normal in all even bases $\leq b$ simultaneously. In both cases, we give an explicit construction of a countable family of such numbers.*

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