Computing absolutely normal numbers

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Representation of real numbers

A *base* is an integer b greater than or equal to 2.

The expansion of a real number x in base b is a sequence $a_1 a_2 a_3 \ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

and the sequence $a_1 a_2 a_3 \ldots$ does not end with a tail of $b-1$.

Normal numbers

Normality is the most basic form of randomness for real numbers.
It was defined by Émile Borel in 1909.

Normal numbers

Definition (Borel, 1909)

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A real x is absolutely normal if x is normal to every base. Hence, a real x is absolutely normal if it is simply normal to all bases b.

Equivalently, x is normal to base b if every block of digits occurs in the expansion of x in base b with limiting frequency equal to $1/b^k$, where k is the block length.

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- Stoneham number $\alpha_{2,3} = \sum_{k\geq 1} \frac{1}{3^k\ 2^{3^k}}$ is normal to base 2 but not simply normal to base 6 (Bailey, Borwein, 2012).

Absolutely normal numbers

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He asked for one example.

Absolutely normal numbers

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Absolutely normal, non-computable constructions

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q [1] tout nombre réel x dont la partie fractionnaire

[1] E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Computable absolutely normal

Definition (Turing 1936)

A real number x is computable if there is a program that produces the expansion of x in some base.

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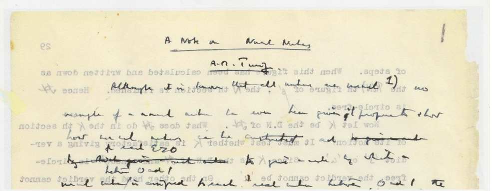
Theorem (Turing 1937?)

There is a computable absolutely normal number.

Turing's algorithm has exponential time complexity: to produce the n-th digit in the expansion of x in base 2 it performs a number of operations that is exponential in n.

Corrected and completed in Becher, Figueira and Picchi, 2007.

Alan Turing, A note on normal numbers, 1937? Collected Works, Pure Mathematics, J.L.Britton ed.1992. 117-119. Notes of editor J.L. Britton, 263-265. North Holland, 1992.



A Note on Normal Numbers

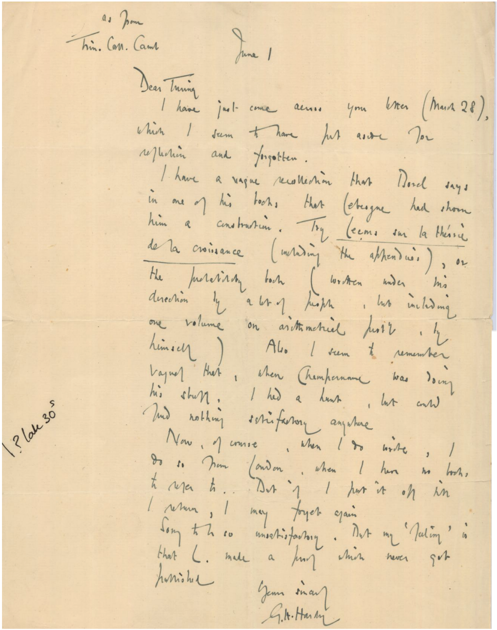
Although it is known that almost all numbers are normal 1) no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R} t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible 2) to prove that

Letter exchange between Turing and Hardy (AMT/D/5)



June 1

Dear Turing,

I have just came across your letter (March 28) which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction.
Try Leçons sur la théorie de la croissance (including the appendices), or the productivity book (written under his direction by a lot of people, but including one volume on arithmetical prosy, by himself).

Also I seem to remember vaguely that when Champernowne was doing his stuff I had a hunt, but could not find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till my return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that Lebesgue made a proof which never got published.

Yours sincerely,

G.H. Hardy

General construction of a computable real number

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Turing uses dyadic intervals. To determine $I_1, I_2, I_3 \ldots$ his strategy is to "follow the measure". The computed number is the trace of left/right choices.

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Equivalenlty,

$$\forall \varepsilon \; \exists n_0 \; \forall n \geq n_0 \left| \frac{|a_1 \ldots a_n|_d}{n} - \frac{1}{b} \right| < \varepsilon.$$

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$$Bad(n) = \bigcup_{b=2}^{t_n} \bigcup_{d=0}^{b-1} \bigcup_{n=n_0(b)}^{N_n} \left\{ x \in (0,1) : \left| \frac{|a_1 \ldots a_n|_d}{n} - \frac{1}{b} \right| \geq \varepsilon_n \right\}$$

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The set

$$\bigcup_{n \geq 1} Bad(n)$$

includes all non normal numbers. It can be proved that it has small measure.

Computing absolutely normal numbers

Define $I_0, I_1, I_2 \ldots$ such that $|I_n| = 2^{-n}$.

Initial step: $I_0 = (0, 1)$

Inductive step n: Divide I_{n-1} in two halves, I_{n-1}^{left} and I_{n-1}^{right}.

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Otherwise let $I_n = I_{n-1}^{\text{right}}$

The computed real number x is the trace of the left/right choice at each step.

$$x \in \bigcap_{n \geq 0} I_n \qquad \text{and} \qquad x \notin \bigcup_{n \geq 1} Bad(n)$$

Then x is absolutely normal.

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Lutz and Mayordomo (2016) gave an algorithm with nearly linear time.

Speed of convergence to normality

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Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k\geq 0}$ is uniformly distributed modulo one for Lebesgue measure.

Normality as uniform distribution modulo 1

For a sequence $(x_k)_{k\geq 1}$ of real numbers in the unit interval the discrepancy of its first N terms is

$$D_N((x_k)_{k\geq 1}) = \sup_{0\leq u<v\leq 1} \left| \frac{\#\{k : 1 \leq k \leq N \text{ and } u \leq x_k < v\}}{N} - (v-u) \right|.$$

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A sequence $(x_k)_{k\geq 1}$ of real numbers in the unit interval is uniformly distributed if

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Schmidt (1974) showed that for every $(x_k)_{k\geq 1}$ there are infinitely many N such that

$$D_N((x_k)_{k\geq 1}) \geq \frac{\log N}{100N}.$$

Some Van der Corput sequences achieve this discrepancy bound.

The discrepancy estimate of normal numbers

Thus, a real x is normal to base b if and only if $(b^k x)_{k\geq 1}$ is u.d. modulo 1.
Hence, writing $\{x\} = x - \lfloor x \rfloor$, a real x is normal to base b if and only if

$$\lim_{N\to\infty} D_N(\{b^k x\}_{k\geq 0}) = 0.$$

The discrepancy estimate of almost all normal numbers

Theorem (Gál and Gál 1964; Philipp 1975, Fukuyama 2008)

Let θ be a real greater that 1.
For almost all reals x, there is $N_0(\theta)$ such that for all greater N,

$$D_N(\{\theta^k x\}_{k\geq 0}) < C_\theta \frac{\sqrt{\log\log N}}{\sqrt{N}},$$

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In case θ is an integer greater than or equal to 2,

$$C_\theta = \begin{cases} \sqrt{84}/9, & \text{if } \theta = 2 \\ \sqrt{2(\theta+1)/(\theta-1)}/2, & \text{if } \theta \text{ is odd} \\ \sqrt{2(\theta+1)\theta(\theta-2)/(\theta-1)^3}/2, & \text{if } \theta \geq 4 \text{ is even.} \end{cases}$$

An instance with discrepancy below the average

Theorem (Aistleitner, Becher, Scheerer and Slaman 2017)

There is an algorithm to compute a real x such that for each integer $b \geq 2$ there is $N_0(b)$ such that for every $N \geq N_0(b)$,

$$D_N(\{b^k x\}_{k\geq 0}) < \frac{C_b}{\sqrt{N}}.$$

For the constant C_b we can take $3433\ b$.

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The algorithm computes the first n digits of the expansion of x in base 2 after exponential in n mathematical operations.

It was not known that such an instance existed.

Normal numbers and their discrepancy estimate

For just one base b Levin 1999 constructed a real x such that

$$D_N(\{b^k x\}_{k\geq 0}) \text{ is } O\left(\frac{\log^2(N)}{N}\right).$$

Open questions

- Asked by Korobov (1955): For a *fixed* integer $b \geq 2$, what is the function $\psi(N)$ with maximal speed of decrease to zero such that there is a real number x for which

$$D_N(\{b^k x\}_{k\geq 0}) = \mathcal{O}(\psi(N)) \qquad \text{as } N \to \infty?$$

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- Is it possible to construct one instance in polynomial time?
- Are there Martin-Löf random with minimal asymptotic $D_N(\{b^k x\}_{k\geq 0})$?

Normality to different bases

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Not perfect powers $\{2, 3, 5, 6, 7, 10, 11, \ldots\}$ are pairwise mutually independent.

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Two integers are multiplicatively dependent if one is a rational power of the other. Not perfect powers $\{2, 3, 5, 6, 7, 10, 11, \ldots\}$ are pairwise mutually independent.

Theorem (Cassels 1959; Schmidt 1961/1962; Becher, Slaman 2013)

For any subset S of the multiplicative dependence classes, there is a real x which is normal to the bases in S and not simply normal to the bases in the complement of S. Furthermore, the real x is computable from S.

Simple normality to different bases

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If x is simply normal to b^m and $\ell | m$ then x is simply normal to b^ℓ.

If x is simply normal to infinitely many powers of b then x is simply normal to all powers of b.

Simple normality to different bases

Theorem (Becher, Bugeaud, Slaman 2013)

Let f be any function from the multiplicative dependence classes to their subsets such that

- *for each b, if $b^{km} \in f(b)$ then $b^k \in f(b)$*
- *if $f(b)$ is infinite then $f(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x which is simply normal to exactly the bases specified by f. Furthermore, the real x is computable from the function f.

The theorem gives a complete characterization (necessary and sufficient conditions).

Absolutely normal and continued fraction normal

For a real number x in the unit interval, the continued fraction expansion of x is a sequence of positive integers $a_1, a_2, \ldots$, such that

$$x = \lfloor x \rfloor + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}$$

we write $[\lfloor x \rfloor; a_1, a_2, \ldots]$, or simply, $[a_1, a_2, \ldots]$ in case x is in the unit interval.

Absolutely normal and continued fraction normal

The Gauss map T is a function from real numbers in the unit interval to real numbers in the unit interval defined by

$$T(0) = 0 \qquad \text{and} \qquad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

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If $[a_1, a_2, \ldots]$ denotes the continued fraction expansion of x, then $T^n(x) = [a_{n+1}, a_{n+2}, \ldots]$ and $a_n = \lfloor 1/T^{n-1}(x) \rfloor$, for $n \geq 1$.

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The Gauss map T is a function from real numbers in the unit interval to real numbers in the unit interval defined by

$$T(0) = 0 \qquad \text{and} \qquad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

If $[a_1, a_2, \ldots]$ denotes the continued fraction expansion of x, then $T^n(x) = [a_{n+1}, a_{n+2}, \ldots]$ and $a_n = \lfloor 1/T^{n-1}(x) \rfloor$, for $n \geq 1$.

The map T possesses an invariant ergodic measure, the Gauss measure μ, which is absolutely continuous with respect to Lebesgue measure; for a Lebesgue measurable set A,

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} \, dx.$$

where $\log$ denotes the logarithm in base e.

Absolutely normal and continued fraction normal

Definition

A real number is continued fraction normal if every block of integers occurs in the continued fraction expansion with the asymptotic frequency determined by the Gauss measure.

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The continued fraction of e is $[2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$. It is the concatenation of the pattern $(1m1)$, for all even m in increasing order, hence not continued fraction normal.

Absolutely normal and continued fraction normal

Theorem (Scheerer 2016; Becher and Yuhjtman 2017)

There is an algorithm that computes a number that is absolutely normal and continued fraction normal.

Exponential time algorithm, Scheerer (2016);
Polynomial time $O(n^4)$, Becher and Yuhjtman (2017).

Normality together with other properties

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ is u.d. modulo 1 for Lebesgue measure.

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For non-zero dimensional sets, Hochman and Shmerkin (2015) give geometrical conditions on a measure μ so that μ-almost all numbers are normal to a given base.

Measures whose Fourier transform decays quickly

A real x is normal to base b iff $(b^k x)_{k\geq 1}$ is u.d. modulo 1 iff (Weyl's criterion) for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

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Lemma (application of Davenport, Erdős, LeVeque's Theorem, 1963)

Let μ be a measure whose Fourier transform decays quickly, let I be an interval and let b a base. If for every non-zero integer t,

$$\sum_{n\geq 1} \frac{1}{n} \int_I \left| \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \; < \infty$$

then for μ-almost all x in interval I are normal to base b.

Absolutely normal Liouville numbers

Kaufman (1981) defined for each a greater than 2, a measure on Jarník's fractal for a whose Fourier transform decays quickly.

Bluhm (2000) defined a measure such that it is supported by the Liouville numbers and its Fourier transform decays quickly.

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Theorem (Becher, Heiber, Slaman 2014)

There is a computable absolutely normal Liouville number.

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The End

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