Constructing normal numbers

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This research originates in a problem posed more than 100 years ago. To a large extent, the problem is still open.

In 1909 Émile Borel gave a definition of the most elementary form of randomness for a real number, thinking in the sequence of digits that determine its expansion.

Randomness - aléatoire - Zufall - azar - rasgelelik - satunnaisuuden - slumpmässighet

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He posed the problem: Give an example.

Normal numbers

A base is an integer greater than or equal to 2.

For a real number x in the unit interval, the expansion of x in base b is a sequence $a_1 a_2 a_3 \ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = 0.a_1 a_2 a_3 \ldots$$

where $x = \sum_{k \geq 1} \frac{a_k}{b^k}$, and x does not end with a tail of $b-1$.

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A real number x is simply normal to base b if, in the expansion of x in base b, each digit occurs with limiting frequency equal to $1/b$.

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Equivalently: a real number x is normal to base b if, for every positive integer k, x is simply normal to base b^k.

A real number x is absolutely normal if x is normal to every base.

Not normal

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

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The rational numbers are not normal to any base.

Liouville's constant $\sum_{n\geq 1} 10^{-n!}$ is not normal to any base.

Existence

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The set of absolutely normal numbers in the unit interval has Lebesgue measure 1.

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Are the usual mathematical constants, such as π, e, or $\sqrt{2}$, absolutely normal? Or at least simply normal to some base?

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Are the usual mathematical constants, such as π, e, or $\sqrt{2}$, absolutely normal? Or at least simply normal to some base?

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Constructions based on concatenation

Normal to a given base

Theorem (Champernowne, 1933)

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	base 10	*base* 3
$x =$	$(0.25)_{10} =$	$(0.020202020202\ldots)_3$
$y =$	$(0.0017)_{10} =$	$(0.000001020110110010 2\ldots)_3$
$x + y =$	$(0.2517)_{10}$	

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$y =$	$(0.0017)_{10} =$	$(0.0000010201101100102\ldots)_3$
$x + y =$	$(0.2517)_{10} =$	$(0.0202101110122\ldots)_3$

Constructions based on discrete counting

Normal to all bases, non-computable constructions

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

General construction of a computable real number

Consider a computable sequence $(I)_{i\geq 1}$ of non-empty intervals I_i with rational endpoints (left endpoint increasing, right endpoints decreasing), nested, length goes to zero.

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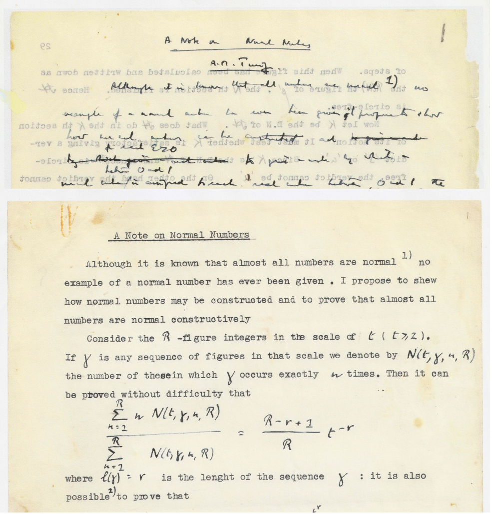
Normal to all bases, computable-construction

Alan Turing, A note on normal numbers, 1937? Collected Works, Pure Mathematics, J.L.Britton ed.1992.



A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible [2] to prove that

Corrected and completed in Becher, Figueira and Picchi, 2007.

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> *"It may also be natural that an example of [an absolutely] normal number be demonstrated as such and written down.*
>
> *This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers but rather, as a counter [...] that the existence proof of normal numbers provides no example of them.*
>
> *The arguments in this note, in fact, follow the existence proof fairly closely."*

Turing's algorithm for computing normal numbers

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Turing gives the following construction. For each k, n,

- $E_{k,n}$ is a finite union of open intervals with rational endpoints.
- Lebesgue measure of $E_{k,n}$ is equal to $1 - \frac{1}{k} + \frac{1}{k+n}$.
- $E_{k,n+1} \subset E_{k,n}$.

For each k, the set $\bigcap_n E_{k,n}$ has Lebesgue measure exactly $1 - \frac{1}{k}$ and consists entirely of absolutely normal numbers.

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There is an algorithm that, given an integer k and an infinite sequence ν of zeros and ones, produces an absolutely normal number $\alpha(k, \nu)$ in the unit interval, expressed in base two.

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At each step, divide the current interval in two halves.

Choose the half that includes normal numbers in large-enough measure
(at step n, intersect half of the current interval with $E_{k,n}$.)
If both halves do, use the current bit of the sequence ν to decide.

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The output $\alpha(k,\nu)$ is the trace of the left/right selection at each step.

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Schmidt 1961/1962, Becher and Figueira 2002 gave other algorithms with exponential complexity.

Normal to all bases, in polynomial time

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For any computable non-decreasing unbounded function f, there is an algorithm that outputs the first n digits in the expansion of a real number in base 2 after $O\big(f(n)n^2\big)$ elementary operations.

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The algorithm is based on Turing's. Speed is gained by

- testing the segment to be added instead of the whole initial segment.
- slowing convergence to normality.

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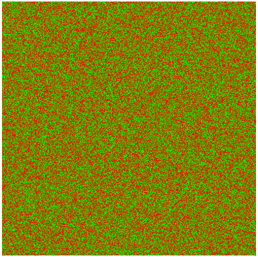
Lutz and Mayordomo (2013) and Figueira and Nies (2013) have another argument to compute an absolutely normal number in polynomial time, based on martingales.

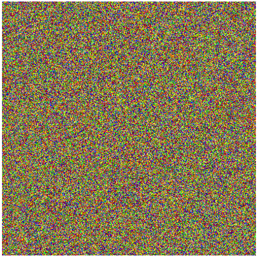
Normal to all bases, in polynomial time

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

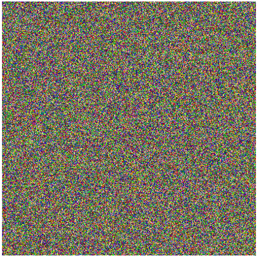
$0.40312905420038091323714283808270591027651167776241897751108966366\ldots$



base 2



base 6



base10

Plots of the first 250000 digits of the output of our algorithm.

Available from `http://www.dc.uba.ar/people/profesores/becher/software/ann.zip`

Open question

Is there an absolutely normal number computable in polynomial time having a nearly optimal rate of convergence to normality?

Constructions based on harmonic analysis

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ is uniformly distributed modulo one for Lebesgue measure.

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This means that the elements

$$\begin{aligned} \{b^0 x\} &= 0.b_1 b_2 b_3 b_4 b_5 \ldots \\ \{b^1 x\} &= 0.b_2 b_3 b_4 b_5 \ldots \\ \{b^2 x\} &= 0.b_3 b_4 b_5 \ldots \\ \{b^3 x\} &= 0.b_4 b_5 \ldots \\ &\vdots \end{aligned}$$

are uniformly distributed in the unit interval.

Normality and Weyl's criterion

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Wall's Theorem

A number x is normal to base b if and only if for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

Normality to different bases

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Theorem (Maxfield 1953)

Let b and b' multiplicatively dependent. For any real number x, x is normal to base b if and only if x is normal to base b'.

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Theorem (Schmidt 1961/1962)

For any given set S of bases closed under multiplicative dependence, there are real numbers normal to every base in S and not normal to any base in its complement. Furthermore, there is a real x computable from S.

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Pollington 1981 showed the set of such numbers has full Hausdorff dimension.
Becher and Slaman 2014 improved the second statement to simple normality, a question of Brown, Moran and Pearce 1988.

Also Levin 1977, reconsidered Alvarez and Becher 2015.

Simple normality to different bases

Fact

If k is a multiple of ℓ, simple normality to b^k implies simple normality to b^ℓ.

Theorem (Long 1957)

Simple normality to infinitely many powers of b implies normality to base b.

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Necessary and sufficient conditions for a set S so that there exists a number that is simply normal to each of the bases in S and not simply normal to each of the bases in the complement of S.

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Moreover, for each such set S, the set of numbers with this condition has full Hausdorff dimension.

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Moreover, for each such set S, the set of numbers with this condition has full Hausdorff dimension.

Also, the asserted real number is computable from the set S.

Uniform distribution modulo one for appropriate measures

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Belief

If we consider appropriate measures, most elements of well structured sets are absolutely normal, unless the sets have evident obstacles.

Appropriate measures for normality

Lemma (direct application of Davenport, Erdős, LeVeque's Theorem 1963)

If μ is a measure on the real numbers such that its Fourier transform vanishes at infinity sufficiently quickly then μ-almost every real number is absolutely normal.

Irrationality exponent

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs (p, q) with positive q.

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- Irrational algebraic numbers have i.e. equal to 2.
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Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs (p, q) with positive q.

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Cantor-like fractals, measures and approximations

- Jarník (1929) and Besicovich (1934) defined a Cantor-like set for reals with a given irrationality exponent.
- Kaufman (1981) defined a measure on Jarník's set whose Fourier transform decays quickly.
- Bluhm (2000) refined it into a measure supported by the Liouville numbers, whose Fourier transform decays quickly.

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Absolute normality and irrationality exponents

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For every real a greater than or equal to 2, there is an absolutely normal number computable in a and with irrationality exponent equal to a.

Simple normality and irrationality exponents

Theorem (Becher, Bugeaud and Slaman, in progress)

Let S be a set of bases satisfying the conditions for simple normality.

- *There is a Liouville number x simply normal to exactly the bases in S.*
- *For every real a greater than or equal to 2 there is a real x with irrationality exponent equal to a and simply normal to exactly the bases in S.*

Furthermore, x is computable from S and, for non- Liouville, also from a.

This theorem is the strongest possible generalization.

Open question

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Which sets admit an appropriate measure for normality?

Hochman and Shmerkin (2015) give a fractal-geometric condition for a measure on $[0, 1]$ to be supported on points that are normal to a given base. This support should have Lebesgue measure 1

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Based on concatenation of prescribed blocks

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2015	(Simply) normal to bases and irrationality exponents	BHS,BBS
Stoneham series (not in this talk)		
1973	Normal to a given base.	Stoneham, Korobov
2012	Normal to base 2 but not to base 6	Bailey and Borwein

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The End

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