Turing's Normal Numbers

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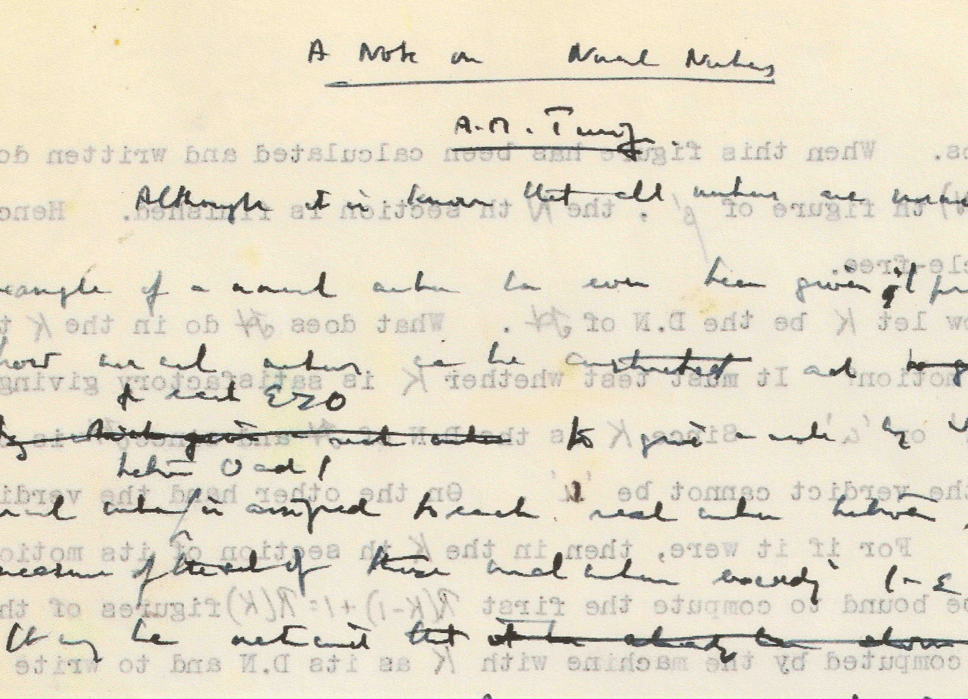
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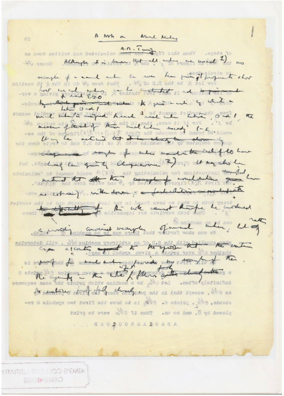
A. M. Turing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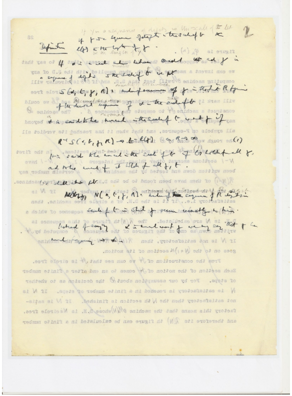
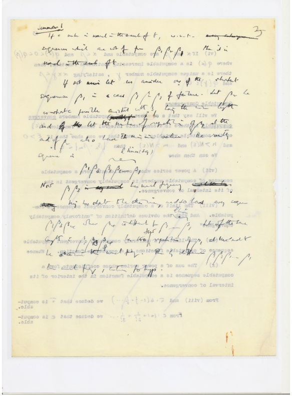
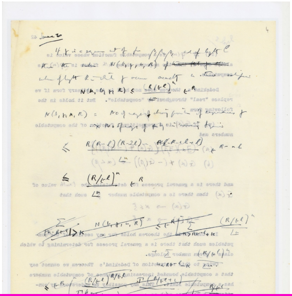
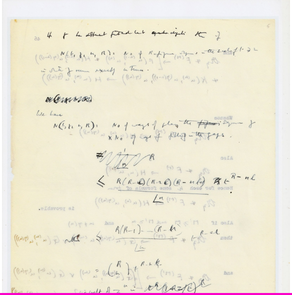
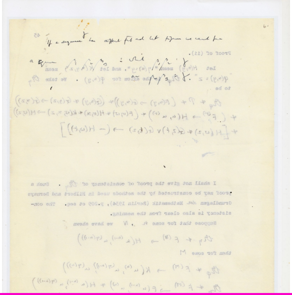






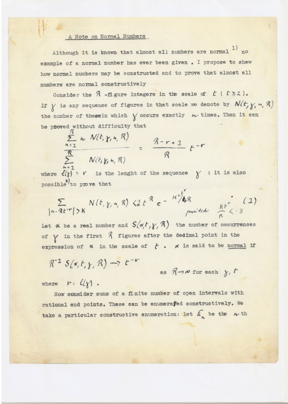
A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t, \gamma,$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also



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where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible [2] to prove that

$$\sum_{|n-Rt^{-r}|>K} N(t,\gamma,n,R) < 2t^{R} e^{-K^2 t^r/4R} \quad (1)$$

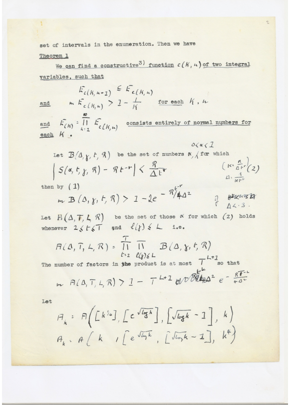
provided $\frac{Kt^r}{R} < \cdot 3$

Let α be a real number and $S(\alpha,t,\gamma,R)$ the number of occurrences of γ in the first R figures after the decimal point in the expression of α in the scale of t. α is said to be normal if

$$R^{-1} S(\alpha,t,\gamma,R) \to t^{-r} \quad \text{as } R\to\infty \text{ for each } \gamma, t$$

where $r = \ell(\gamma)$.

Now consider sums of a finite number of open intervals with rational end points. These can be enumerated constructively. We take a particular constructive enumeration: let E_n be the nth



set of intervals in the enumeration. Then we have

Theorem 1

We can find a constructive [3] function $c(K,n)$ of two integral variables, such that

$E_{c(K,n+1)} \subseteq E_{c(K,n)}$

and $m E_{c(K,n)} > 1 - \frac{1}{K}$ for each K, n

and $E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists entirely of normal numbers for each K.

Let $B(\Delta,\gamma,t,R)$ be the set of numbers α, $0<\alpha<1$, for which

$$\left| S(\alpha,t,\gamma,R) - Rt^{-r} \right| < \frac{R}{\Delta t^r} \quad (2)$$

then by (1)

$$m\, B(\Delta,\gamma,t,R) > 1 - 2e^{-R t^{-r}/4\Delta^2}$$

Let $A(\Delta,T,L,R)$ be the set of those α for which (2) holds whenever $2 \leq t \leq T$ and $\ell(\gamma) \leq L$ i.e.

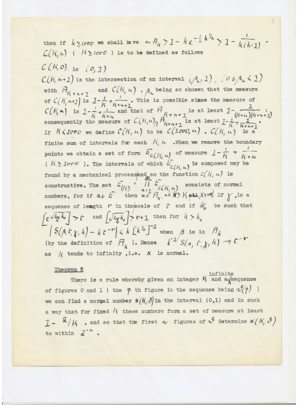
$$A(\Delta,T,L,R) = \prod_{t=2}^{T} \prod_{\ell(\gamma)\leq L} B(\Delta,\gamma,t,R)$$

The number of factors in the product is at most T^{L+1} so that

$$m\, A(\Delta,T,L,R) > 1 - T^{L+1} \cdot 2 e^{-\frac{RT^{-L}}{4\Delta^2}}$$

Let

$$A_k = A\left([k^{1/4}], [e^{\sqrt{\log k}}], [\sqrt{\log k} - 1], k\right)$$



then if $k \geq 1000$ we shall have $m A_k > 1 - ke^{-\frac{1}{2}k^{1/2}} > 1 - \frac{1}{k(k-1)}$.

$C(K,n)$ ($K \geq 1000$) is to be defined as follows

$C(K,0)$ is $(0,1)$

$C(K,n+1)$ is the intersection of an interval $(\beta_n, 1)$, $(0 \leq \beta_n < 1)$ with A_{K+n+1} and $C(K,n)$, β_n being so chosen that the measure of $C(K,n+1)$ is $1-\frac{1}{K}+\frac{1}{K+n+1}$. This is possible since the measure of $C(K,n)$ is $1-\frac{1}{K}+\frac{1}{K+n}$ and that of A_{K+n+1} is at least $1-\frac{1}{(K+n)(K+n+1)}$; consequently the measure of $C(K,n) \cap A_{K+n+1}$ is at least $1-\frac{1}{K}+\frac{1}{K+n+1}$.

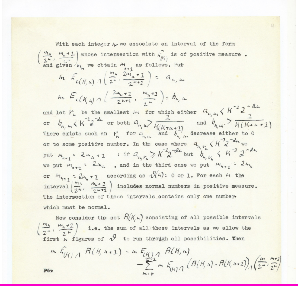
If $K < 1000$ we define $C(K,n)$ to be $C(1000,n)$. $C(K,n)$ is a finite sum of intervals for each K, n. When we remove the boundary points we obtain a set of form $E_{c(K,n)}$ of measure $1-\frac{1}{K}+\frac{1}{K+n}$ ($K \geq 1000$). The intervals of which $E_{c(K,n)}$ is composed may be found by a mechanical process, so the function $c(K,n)$ is constructive. The set $E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists of normal numbers, for if $\alpha \in E$ then $\alpha \in A_k$ all $k > K$. If γ is a sequence of length r in the scale of t and if k_0 be such that $[e^{\sqrt{\log k_0}}] > t$ and $[\sqrt{\log k_0}] > r+1$ then for $k > k_0$

$$\left| S(\beta,t,\gamma,k) - kt^{-r} \right| < k[k^{1/4}]^{-1} \quad \text{when } \beta \text{ is in } A_k$$

(by the definition of A_k). Hence $k^{-1} S(\alpha,t,\gamma,k) \to t^{-r}$ as k tends to infinity, i.e. α is normal.

Theorem 2

There is a rule whereby given an integer K and an infinite sequence of figures 0 and 1 (the p th figure in the sequence being $\vartheta(p)$) we can find a normal number $\alpha(K,\vartheta)$ in the interval (0,1) and in such a way that for fixed K these numbers form a set of measure at least $1-2/K$, and so that the first n figures of ϑ determine $\alpha(K,\vartheta)$ to within 2^{-n}.



With each integer n we associate an interval of the form $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ whose intersection with $E_{(K)}$ is of positive measure, and given m_n we obtain m_{n+1} as follows. Put

$$m\, E_{c(K,n)} \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n,m}$$

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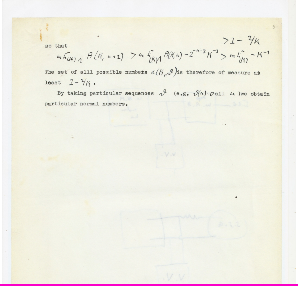
and let r_n be the smallest m for which either $a_{n,m} < K^{-1}2^{-2n}$ or $b_{n,m} < K^{-1}2^{-2n}$ or both $a_{n,m} > \frac{1}{K(K+n+1)}$ and $b_{n,m} > \frac{1}{K(K+n+1)}$. There exists such an r_n for $a_{n,m}$ and $b_{n,m}$ decrease either to 0 or to some positive number. In the case where $a_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n+1$: if $a_{n,r_n} \geq K^{-1}2^{-2n}$ but $b_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n$, and in the third case we put $m_{n+1} = 2m_n$ or $m_{n+1} = 2m_n+1$ according as $\vartheta(n) = 0$ or 1. For each n the interval $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ includes normal numbers in positive measure. The intersection of these intervals contains only one number which must be normal.

Now consider the set $A(K,n)$ consisting of all possible intervals $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ i.e. the sum of all these intervals as we allow the first n figures of ϑ to run through all possibilities. Then

$$m\, E_{(K)} \cap A(K,n+1) = m\, E_{(K)} \cap A(K,n) - \sum_{m=0}^{2^n-1} m\, E_{(K)} \cap (A(K,n) - A(K,n+1)) \cap \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$$

For



so that

$$m E_{(K)} \cap A(K,n+1) > m E_{(K)} \cap A(K,n) - 2^{-n-1}K^{-1} > m E_{(K)} - K^{-1} > 1 - 2/K$$

The set of all possible numbers $\alpha(K,\vartheta)$ is therefore of measure at least $1-2/K$.

By taking particular sequences ϑ (e.g. $\vartheta(n)=0$ all n) we obtain particular normal numbers.

A Note on Normal Numbers, Alan M. Turing

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Reconstructed, corrected and completed in 2007
Becher, Figueira, Picchi, *Theoretical Computer Science* 377, 126-138.

Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

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For instance, if a number is normal to base 2, each of the digits '0' and '1' occur in the limit, half of the times; each of the blocks '00', '01', '10' and '11' occur one fourth of the times, and so on.

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A real number is *normal to a given integer base* if its expansion in that base is evenly balanced: every block of digits of the same length occurs with the same limit frequency.

A real number that is normal to every integer base is called *absolutely normal*, or just *normal*.

Counterexamples

0.1010010001000010000010000... not normal to base 2.

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Rationals are not normal (for each $q \in \mathbb{Q}$ there is a base b such that the expansion of q ends with all zeros).

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Theorem (Borel 1909)

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Borel asked for an explicit example.

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$$\sum N(t,\gamma,n,R) < 2t^{R} e^{-k^2 t^r / 4R} \qquad (1)$$

Turing's Note on Normal Numbers

Turing's Theorem 1

Borel's theorem on the measure of normal numbers, constructively.

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Turing's Theorem 2

An algorithm to construct normal numbers.

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Turing's First Page of the Handwritten Manuscript

His own appraisal of his work.

Turing's Theorem 1

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We can find a constructive[3] function $c(K, n)$ of two integral variables, such that

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and $E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists entirely of normal numbers for each K.

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$c(k, n)$ is such that

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$c(k, n)$ is such that

- $E_{c(k,n)}$ is included in $E_{c(k,n-1)}$ and
- measure of $E_{c(k,n)}$ is greater than $1 - 1/k$.

Finally, for each k, $E(k) = \bigcap_n E_{c(k,n)}$ has measure exactly $1 - 1/k$ and it consists entirely of normal numbers.

Main idea in Turing's Theorem 1: finite approximations

The construction is uniform in the parameter k.

Prune the unit interval, by stages.

Stage 0: $E_{c(k,0)}$ is the whole unit interval.

Stage n: $E_{c(k,n)}$ results from removing from $E_{c(k,n-1)}$ the points that are **not** candidates to be normal, according to the inspection of an initial segment of their expansions.

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At the end, the construction discards

- all rational numbers, because of their periodic structure
- all irrational numbers with an unbalanced expansion
- all normal numbers whose convergence to normality is too slow

$E(k) = \bigcap_n E_{c(k,n)}$ consists entirely of normal numbers.

Its measure is exactly $1 - 1/k$ (because $E_{c(k,n)}$ measures $1 - \frac{1}{k} + \frac{1}{k+n}$).

Main idea in Turing's Theorem 1: finite approximations

Computable functions of the stage n,

initial segment size	linear
base	sublinear
block length	sublogarithmic
frequency discrepancy ...	the technically largest converging to 0

$E_{c(k,n)}$, the set of reals not discarded up to stage n, is the union of finitely many intervals, tailored to measure $1 - \frac{1}{k} + \frac{1}{k+n}$.

Constructive Strong Law of Large Numbers

In most initial segments:

each single digit occurs about the expected number of times
each block of two digits occurs about the expected number of times
...
each block short-enough occurs about the expected number of times.

Lemma (extends Hardy & Wright 1938)

Fix b, w of length ℓ and N. For any real ε such that $\frac{7}{N} \leq \varepsilon \leq \frac{1}{b^\ell}$,

$$\sum_{\left|\frac{i}{N}-\frac{1}{b^\ell}\right|\geq\varepsilon} \begin{array}{l}\textit{number of blocks of length } N \\ \textit{with exactly } i \textit{ occurrences of } w\end{array} \leq b^N \, 2 \, b^{2\ell} e^{-\frac{b^\ell \varepsilon^2 N}{6\ell}}.$$

Turing's Theorem 2

Theorem 2

There is a rule whereby given an integer K and an infinite sequence of figures 0 and 1 (the p th figure in the sequence being $\vartheta(p)$) we can find a normal number $\alpha(K,\vartheta)$ in the interval (0,1) and in such a way that for fixed K these numbers form a set of measure at least $1 - 2/K$, and so that the first n figures of ϑ determine $\alpha(K,\vartheta)$ to within 2^{-n} .

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In order to write down the first n digits of $\alpha(k, \nu)$ the algorithm requires at most the first n digits of ν.

For a fixed k these numbers $\alpha(k, \nu)$ form a set of measure at least $1 - 2/k$.

The idea in Theorem 2: “follow the measure”

It works by steps.

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At each step, divide the current interval in two halves, and choose the half that includes normal numbers in large-enough measure.

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It works by steps.

Start with the unit interval.

At each step, divide the current interval in two halves, and choose the half that includes normal numbers in large-enough measure.

If both halves do, use the current bit of the oracle to decide
(this will happen infinitely often)

The output $\alpha(k, \nu)$ is the trace of the left/right selection at each step.

Algorithm

With each integer n we associate an interval of the form $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ whose intersection with $E_{(K)}$ is of positive measure. and given m_n we obtain m_{n+1} as follows. Put

$$m\, E_{c(K,n)} \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n,m}$$

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Correctness of the algorithm

- Invariant: $I_n \cap E(k)$ has positive measure.
- Threshold: $M(k,n)$ is a lower bound of $\mu(E_{c(k,n)} \cap I_n)$ verifying

$$M(k,n) = M(k,n-1)/2 - (\mu E_{c(k,n)} - \mu E_{c(k,n+1)})/2.$$

- Output: $\alpha(k,n) = \bigcap_n I_n$, with explicit convergence to normality.

Turing's normal numbers

By taking particular instances of the input sequence ν the set of numbers that can be output has measure at least $1 - 2/k$.

When ν is computable (Turing puts all zeros), the algorithm yields a computable normal number.

The algorithm can be adapted to intercalate the bits of ν at fixed positions of the output sequence.

Theorem (Figueira PhD Thesis 2006)

There is a normal number in each Turing degree.

Computational Complexity of Turing's algorithm

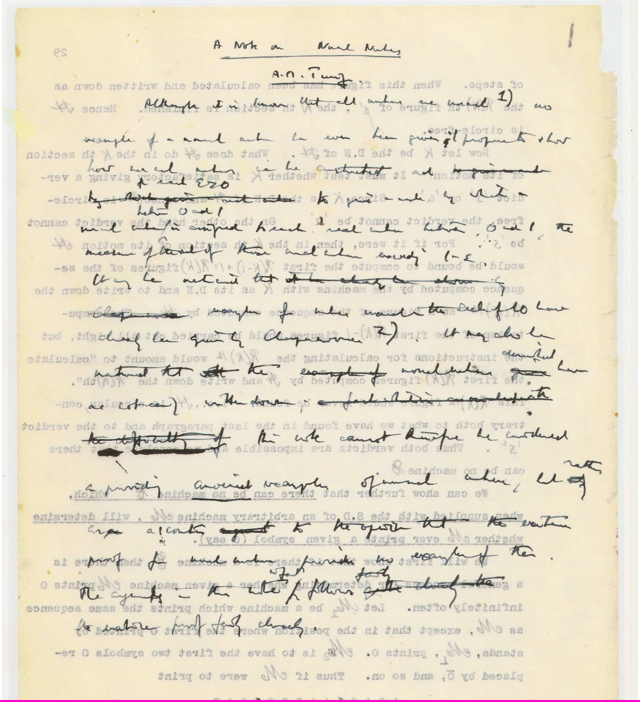
The number of operations to produce a next digit in the output

- *simple-exponentially* many (literal reading)
- *double-exponentially* many (our reconstruction)

Turing's First Page of the Handwritten Manuscript

Not transcribed.

His own appraisal of his work.



A Note on Normal Numbers

A. M. Turing

Although it is known that all numbers are normal 1) no example of a normal number has ever been given. I propose to show how normal numbers can be constructed and ~~to give numbers~~ ~~by [illegible]~~ to give a rule by which, for each ε>0, a normal number is assigned to each real number between 0 and 1, the measure of the set of these normal numbers exceeding 1-ε.

It may be noticed that ~~[illegible]~~ examples of numbers normal in the scale of 10 have ~~[illegible]~~ been given by Champernowne 2). It may also be noticed that ~~the~~ the ~~examples of~~ normal numbers described here are not constructive (written down ~~[illegible]~~)

~~the difficulty~~ of this rule cannot therefore be considered as providing convincing examples of normal numbers, but rather as a ~~counter argument~~ to the opinion that the existence proof for normal numbers provides no example of them.

The argument – the rule for what follows fairly ~~[illegible]~~ the natural proof fairly closely.

Turing's First Page of the Handwritten Manuscript

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415...
as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

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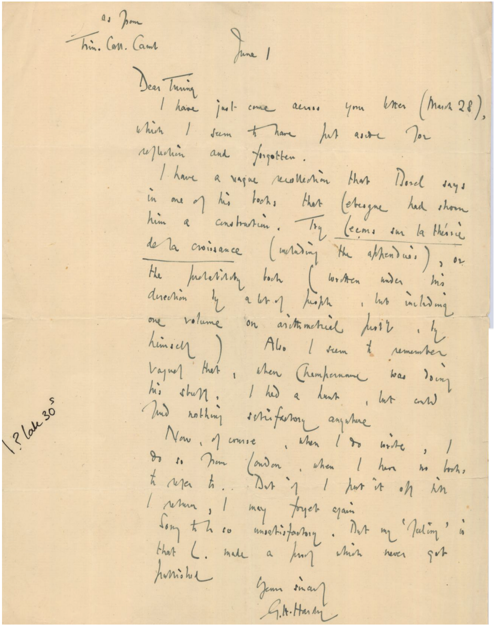
Letter exchange between Turing and Hardy (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob[abilit]y, by himself). Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till I return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

? late 30s

G.H. Hardy was right but he missed the novelty

Henri Lebesgue in 1909

Waclaw Sierpiński in 1916

independently, each gave a non-finitary based construction:

Bulletin de la Société Mathématique de France 45:127–132 and 132–144, 1917

Turing's Note on Normal Numbers

A story of unrecognized scientific priority

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In particular, Turing pioneered the theory of algorithmic randomness.

Turing's Normal Numbers: Towards Randomness

A real is random if it exhibits the almost-everywhere behavior of all reals.
A random real must pass every test of these properties; for instance,
its expansion must be evenly balanced.

Turing's Normal Numbers: Towards Randomness

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Definition (Martin-Löf 1966)

A test for randomness is a uniformly computably enumerable sequence of sets of intervals with rational endpoints whose measure is upper-bounded by a computable function and converges to zero.

A real number is random if it is covered by no such test.

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A real number is random if it is covered by no such test.

Corollary (Randomness Implies Normality)

The sequence $((0,1) \setminus E(k))_{k \geq 0}$ *is a ML-test.*

Surprise

Absolutely normal numbers in just above quadratic time

Theorem (Becher, Heiber, Slaman 2013)

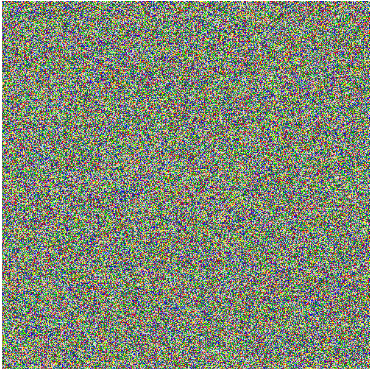
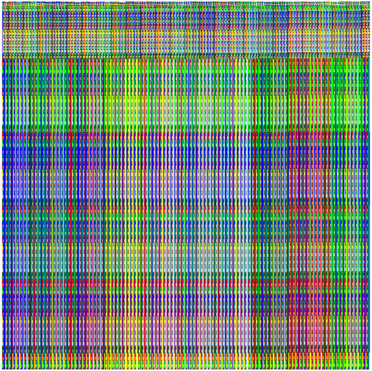
Suppose $f : \mathbb{N} \to \mathbb{N}$ is a computable non-decreasing unbounded function. There is an algorithm to compute an absolutely normal number x such that, for any base b, the algorithm outputs the first n digits in of its expansion after $O(f(n)\, n^2)$ elementary operations.

Lutz, Mayordomo 2013 and also Figueira, Nies 2013 have another argument for an absolutely normal number in polynomial time, based on polynomial-time martingales.

The output of our algorithm in base 10

Programmed by Martin Epszteyn

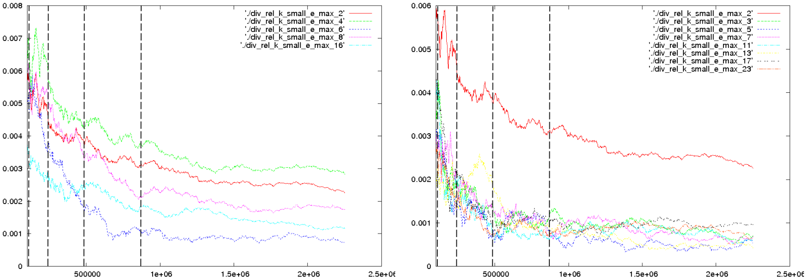
0.4031290542003809132371428380827059102765116777624189775110896366...

	
First 250000 digits output by the algorithm	First 250000 digits of Champernowne
Plotted in 500x500 pixeles, 10 colors	Plotted in 500x500 pixeles, 10 colors

Algorithm with parameters $t_i = (3 * \log(i)) + 3$; $\epsilon_i = 1/t_i$ Initial values $t_1 = 3$; $\epsilon_1 = 1$.

First extension in base 2 is of length $k = 405$. Then k increases only when necessary.

The output of our algorithm in each base



Left: Discrepancy for powers of 2, normalized by expected frequency.
Right: Discrepancy for prime digits, normalized by expected frequency.

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To Cristian Calude for suggesting the problem of Sierpiński's normal numbers to me.

To Gregory Chaitin for pointing out Turing's Note on Nomal Numbers.

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To Alexander Shen for his help with a missing argument in the reconstruction of Turing's Theorem 2.

To Joos Heintz for encouraging me for more than ten years to find a polynomial time algorithm to produce absolutely normal numbers.

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