Normal numbers with digit dependencies

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Expansion of a real number in an integer base

For a real number x, its expansion in an integer base $b \geqslant 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leqslant a_n < b$ for every n, such that

$$x - \lfloor x \rfloor = \sum_{n \geqslant 1} a_n b^{-n} = 0.a_1 a_2 a_3 \ldots$$

We require that $a_n < b - 1$ infinitely often to ensure that every number has a unique representation.

Borel normal numbers

Let integer $b \geqslant 2$. A real number x is simply normal to base b if every digit in $\{0, \ldots, b-1\}$ occurs in the base-b expansion of x with the same asymptotic frequency (that is, with frequency $1/b$).

A real number x is normal to base b if it is simply normal to all the bases $b, b^2, b^3, \ldots$.

A real number x is absolutely normal if it is normal to all integer bases.

Normality to base b is equivalent to normality to any base multiplicatively dependent to b (integers x and y are multiplicatively dependent if there are s, t such that $x^s = y^t$)

Borel proved that almost all numbers, with respect to Lebesgue measure, are absolutely normal.

Examples and counterexamples of Borel normal numbers

$0.01010101010\ldots$ is simply normal to base 2 but not to 2^2 nor 2^3, etc.

Each number in Cantor middle third set is not simply normal to base 3.

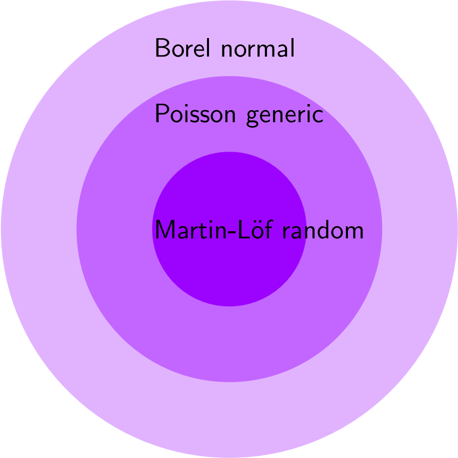
Champernowne's number in base 10,

$$0.12345679101112131415161718192021$$

is normal to base 10, but is not known whether it is normal to bases multiplicatively independent to 10.

Stoneham number $\alpha_{2,3} = \sum_{n \geqslant 1} \frac{1}{3^n \; 2^{3^n}}$ is normal to base 2 but not simply normal to base 6 (Bailey and Borwein, 2012).

Borel normal numbers and other properties of full measure

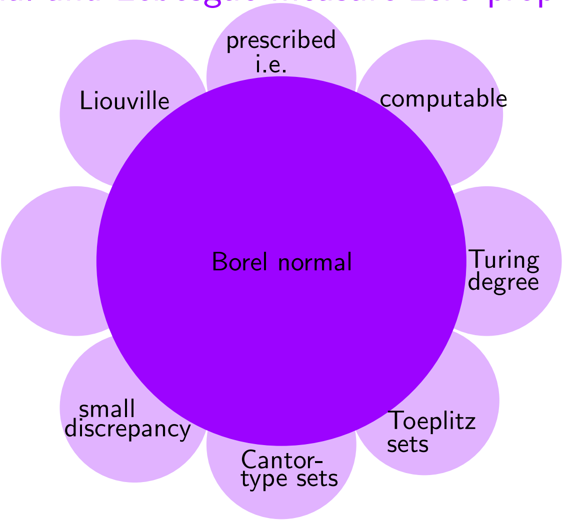


Schnorr (1975);

Rudnick (supernormal), Peres,Weiss (2020),Alvarez, B., Mereb (2022); B., Sac Himelfarb (2022)

Also normality for other numeration systems, such as continued fractions.

Borel normal and Lebesgue measure zero properties



Turing(1937), Cassels (1959), Schmidt (1961/1962), Bugeaud (2002), Levin (1999)

Conjecture (Borel 1951)

All algebraic irrational numbers are absolutely normal.

Metric results on Cantor-type sets

Cassels 1959 worked on a Cantor-type set of real numbers whose ternary expansion avoids the digit 2 (hence not normal to base 3) and uniform measure supported on this Cantor-type set. Schmidt $1961/1962$ generalized this to arbitrary bases b.

They obtain that, relative to the uniform measure in this Cantor-type set for base b, almost all numbers are normal to all the bases are multiplicatively independent to b.

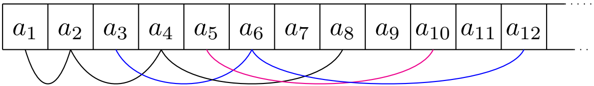
Their proofs give upper bounds for certain Riesz products.

Toeplitz numbers (Jacobs and Keane 1969)

Let integer $b \geqslant 2$ and let $\mathcal{P}$ be a set of prime numbers.

The set of Toeplitz numbers $\mathcal{T}_{b,\mathcal{P}}$ is the set of all real numbers $x \in [0,1)$ such that if $x = \sum_{n \geqslant 1} a_n b^{-n}$ then $a_n = a_{np} \qquad (n \geqslant 1,\, p \in \mathcal{P})$.

For example $0.a_1 a_2 a_3 \dots$ is a Toeplitz number for $\mathcal{P} = \{2\}$ if, for every $n \geqslant 1$, we have $a_n = a_{2n}$,



$a_1 = a_2 = a_4 = a_8 = \dots$ (black)
$a_3 = a_6 = a_{12} = a_{24} = \dots$ (blue)
$a_5 = a_{10} = a_{20} = a_{40} = \dots$ (pink)
The variables with odd indices are free.

Uniform measure on $\mathcal{T}_{b,\mathcal{P}}$

Let integer $b \geqslant 2$ and let $\mathcal{P}$ be a set of prime numbers.

Let $j_1, j_2, \ldots$ be the enumeration in increasing order of all positive integers that are not divisible by any of the primes in $\mathcal{P}$.

The Toeplitz transform $\tau_{b,\mathcal{P}} : [0,1) \to \mathcal{T}_{b,\mathcal{P}}$ is defined as

$$\tau_{b,\mathcal{P}}(0.b_1 b_2 b_3 \ldots) = 0.a_1 a_2 a_3 \ldots$$

such that when $n = j_k p_1^{e_1} \cdots p_r^{e_r} \quad (p_1, \cdots, p_r \in \mathcal{P})$,

$$a_n = b_k.$$

Observe that one can define an equivalence relation $\sim$ between indexes $n \sim n'$ when n and n' have the same factor j that is not divisible by any of the primes $p \in \mathcal{P}$.

Let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$, which is the push-forward of the Lebesgue measure λ by $\tau_{b,\mathcal{P}}$
For measurable $X \subseteq \mathcal{T}_{b,\mathcal{P}}$,

$$\mu(X) = \lambda(\tau_{b,\mathcal{P}}^{-1}(X)).$$

Theorem 1 (Aistleitner, B. and Carton 2019)

Let $b \geqslant 2$, $\mathcal{P} = \{2\}$ and μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to all bases that are multiplicatively independent to b.

Theorem 2 (Aistleitner, B. and Carton 2019)

Let $b \geqslant 2$, let $\mathcal{P}$ be a finite set of primes and let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to base b.

For $\mathcal{P} = \{2\}$ was obtained by Alexander Shen (2016), and by Lingmin Liao and Michal Rams (2021).

Theorems 1 and 2 yield that for $\mathcal{P} = \{2\}$ μ-almost all numbers in $\mathcal{T}_{b,\mathcal{P}}$ are absolutely normal.

Theorem 3 (Aistleitner, B. and Carton 2019)

Let integer $b \geqslant 2$. Let $X_1, X_2, \ldots$ be a sequence of random variables from a given probability space $(\Omega, \mathcal{F}, P)$ into $\{0, .., b-1\}$.

Assume that for every n, X_n is uniformly distributed on $\{0, .., b-1\}$. Suppose there is a function $g: \mathbb{N} \mapsto \mathbb{R}$ unbounded and monotonically increasing such that for all sufficiently large n the random variables

$$X_n,\ X_{n+1},\ \ldots,\ X_{n+\lceil g(n) \log\log n \rceil}$$

are mutually independent. Then, P-almost surely $x = 0.X_1X_2\ldots$ is normal to base b.

Furthermore, $\lceil g(n) \log\log n \rceil$ is sharp.

Theorem 4 (B., Carton and Heiber 2018)

We construct a number in $\mathcal{T}_{b,\mathcal{P}}$ for $b = 2$ and $\mathcal{P} = \{2\}$, normal to base 2.

This is a Champernowne-like construction of a binary expansion such that $a_n = a_{2n}$ for every $n \geqslant 1$. It can be generalized to any base b and any singleton $\mathcal{P}$.

Example of a simply normal number to base b in $\mathcal{T}_{b,\mathcal{P}}$

Let $\mathbb{P}$ be the set of all primes and let $\mathcal{P} \subset \mathbb{P}$.

Define the completely additive arithmetic function $\Omega_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$,

$\Omega_{\mathcal{P}}(n)$ is the total count of prime factors of n that are *not* in $\mathcal{P}$.

For example, for $\mathcal{P} = \{2, 3\}$, $\Omega_{\mathcal{P}}(2^4 \cdot 3^6 \cdot 5^2 \cdot 7) = 3$

Notice that with $\mathcal{P} = \emptyset$ we recover the classical Ω.

Given $\mathcal{P} \subset \mathbb{P}$ and integer $b \geqslant 2$, define the number

$$\xi_{b,\mathcal{P}} := \sum_{n \geqslant 1} t_n b^{-n}, \qquad \text{where } t_n := (\Omega_{\mathcal{P}}(n) \mod b).$$

Clearly, $\xi_{b,\mathcal{P}} \in \mathcal{T}_{b,\mathcal{P}}$.

Theorem 5 (B., Marchionna and Tenenbaum 2023)

Let integer $b \geqslant 2$ and $\mathcal{P} \subset \mathbb{P}$. The number $\xi_{b,\mathcal{P}}$ is simply normal to base b if, and only if, $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p = \infty$. Moreover, defining for $k = 0, \ldots, (b-1)$

$$\varepsilon_{N,k} := \left| \frac{1}{N} \#\{n : 1 \leqslant n \leqslant N, (\Omega_{\mathcal{P}}(n) \mod b) = k\} - \frac{1}{b} \right|$$

we have

$$\varepsilon_{N,k} \ll \frac{1}{b} e^{-E(N)/180b^2}, \text{ where } E(N) := \sum_{p \leqslant N,\, p \in (\mathbb{P} \setminus \mathcal{P})} 1/p \quad (N \geqslant 1).$$

Since $E(N)$ is asymptotyically equal to $\log\log N + M$, where M is Meissel-Mertens constant,

$$\varepsilon_{N,k} \ll \frac{1}{2} e^{-(\log\log N + M)/720} = \frac{1}{2} e^{-M/720} (\log N)^{-1/720}.$$

(Champernowne constant has discrepancy exactly in the order $(\log N)^{-1}$)

Proof of Theorem 5

Ramanujan J.
44, n° 3 (2017), 641-701 ;
Corrig. **51**, n° 1 (2020), 243-244.

Moyennes effectives de fonctions multiplicatives complexes*

Gérald Tenenbaum

Abstract. We establish effective mean-value estimates for a wide class of multiplicative arithmetic functions, thereby providing (essentially optimal) quantitative versions of Wirsing's classical estimates and extending those of Halász. Several applications are derived, including: estimates for the difference of mean-values of so-called pretentious functions, local laws for the distribution of prime factors in an arbitrary set, and weighted distribution of additive functions.

Proof of Theorem 5

Lemma

Let $\mathcal{P} \subset \mathbb{P}$ and let b be an integer $\geqslant 2$. The number $\xi_{b,\mathcal{P}}$ is simply normal to the base b if, and only if,

$$\frac{1}{N}\sum_{n\leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = o(1) \quad (a = 1, 2, \ldots b-1,\ N \to \infty).$$

The necessity of the criterion is clear (the b-th roots of unity have the same asymptotic frequency, and they sum up 0) We show the sufficiency.
Define

$b_{k,N} = \dfrac{1}{N}\#\{1 \leqslant n \leqslant N : (\Omega_{\mathcal{P}}(n) \mod b) = k\},\ (k = 0, \ldots, b-1,\ N \geqslant 1).$

Then

$b_{k,N} = \dfrac{1}{bN}\displaystyle\sum_{0\leqslant a<b} \mathrm{e}(-ak/b)\sum_{n\leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = \dfrac{1}{b} + o(1)$

since

for $a = 0$, the sum is $1/b$

for each term with $a \neq 0$, by hypothesis, all inner sums contribute $o(N)$.

Proof of Theorem 5

In case $\sum_{p\in(\mathbb{P}\setminus\mathcal{P})} 1/p$ diverges:

Lemma (Corolary 2.4 Tenenbaum 2017)

Let $\mathcal{P}\subset\mathbb{P}$, *let* $E(N)=\displaystyle\sum_{p\leqslant N, p\in(\mathbb{P}\setminus\mathcal{P})} 1/p$.
Assume $E(N)\to+\infty$ *when* $N\to+\infty$. *Let* $\kappa\in(0,1)$.
If $\kappa\leqslant r\leqslant 2-\kappa$, $z=re^{i\theta}$, $-\pi\leqslant\theta\leqslant\pi$ *then for every sufficiently large* N,

$$\sum_{n\leqslant N} z^{\Omega_{\mathcal{P}}(n)} \ll N e^{\left(r-1-\frac{\kappa\theta^2}{180}\cdot E(N)\right)}.$$

Proof of Theorem 5

Notice $\{a \in \mathbb{Z} : |a| \leqslant \frac{1}{2}b\}$ describes a complete set of residues $\pmod b$. Whenever a and b are coprime, $b \geqslant 2$ and $|a| \leqslant b/2$, we may apply Corollary 2.4 of T. (lemma above) with $r = 1$, $z = \mathrm{e}(a/b)$, $\vartheta = 2\pi a/b$ and $\kappa = 1$.

$$\sum_{n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) \ll Ne^{-2a^2 E(N)/(9b^2)}.$$

So, if $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p = \infty$ then the above lemma implies that $\xi_{b,\mathcal{P}}$ is simply normal to the base b.

Proof of Theorem 5

In case $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p$ converges:

Lemma (Corolary 2.2 Tenenbaum 2017, effective version of Delange)

Let $A,\ B,\ \varrho$ postive reals . Let complex multiplicative $f : \mathbb{N} \to \mathbb{C}$ such that for every prime p, $|f(p)| \leqslant A \leqslant \varrho$ and $\sum_{p^\nu,\ \nu \geqslant 2} \frac{|f(p^\nu)| \log p^\nu}{p^\nu} \leqslant B$.

If $\sum_p \frac{\varrho - \Re f(p)}{p}$ converges, then

$$\sum_{n \leqslant N} f(n) = \frac{e^{-\gamma \varrho} N}{\Gamma(\varrho) \log N} \left[\prod_{p \in \mathbb{P}} \sum_{p^\nu \leqslant N} \frac{f(p^\nu)}{p^\nu} + \mathcal{O}\left(\nu_N^\alpha e^{Z(N;f)} + \frac{e^{Z(N;f)}}{(\log N)^\beta} \right) \right]$$

where $\alpha := w_f \min(1, \varrho)/(5 - \min(1, \varrho))$, $\beta := w_f \min(1, \varrho)/6$ with $Z(N; f) = \sum_{p \leqslant N} \frac{f(p)}{p}$, and ν_N is such that $\sum_{p \leqslant N} \frac{(\varrho - \Re f(p)) \log p}{p} \ll \nu_N \log N$ with $\nu_N \to 0$ as $N \to \infty$.

Proof of Theorem 5

If $\sum_{p\in(\mathbb{P}\setminus\mathcal{P})} 1/p$ converges by Corollary 2.2 of T. (lemma above) we have

$$\sum_{p\in(\mathbb{P}\setminus\mathcal{P}),\ p\leqslant N} \frac{\log p}{p} \ll \eta_N \log N$$

for some $\eta_N \to 0$. A possible $\eta_N := \min_{1\leqslant z\leqslant N}\left(\frac{\log z}{\log N} + \sum_{p\in(\mathbb{P}\setminus\mathcal{P})\ p>z} \frac{1}{p}\right)$.

To prove that $\sum_{n\leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) \gg N$ it hence suffices to show that taking $f(n) = \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big)$,

$$\sum_{n\leqslant N} f(n) = \frac{e^{-\gamma\varrho}N}{\Gamma(\varrho)\log N}\left[\underbrace{\prod_{p\in\mathbb{P}}\sum_{p^\nu\leqslant N}\frac{f(p^\nu)}{p^\nu}}_{\text{Show}_{\gg \log N}} + \mathcal{O}\left(\nu_N^\alpha e^{Z(N;f)} + \frac{e^{Z(N;f)}}{(\log x)^\beta}\right)\right]$$

Proof of Theorem 5

$$\prod_p \sum_{p^\nu \leqslant N} \frac{\mathrm{e}(\nu a \Omega_{\mathcal{P}}(p)/b)}{p^\nu} = \prod_{p \in (\mathbb{P} \setminus \mathcal{P})} \frac{1 - \mathrm{e}(\nu_p a/b)/p^{\nu_p}}{1 - \mathrm{e}(a/b)/p} \prod_{p \in \mathcal{P}} \frac{1 - 1/p^{\nu_p}}{1 - 1/p} \gg \log N$$

where we have put $\nu_p := 1 + \lfloor (\log N)/\log p \rfloor$, so that $p^{\nu_p} \geqslant N$.
Now the double product above is clearly

$$\sim \sigma_N := \prod_{p \leqslant N} \frac{1}{1 - 1/p} \prod_{p \in (\mathbb{P} \setminus \mathcal{P})} \frac{1 - 1/p}{1 - \mathrm{e}(a/b)/p}.$$

Since the general factor of the last product equals $1 + \{\mathrm{e}(a/b) - 1\}/p + O(1/p^2)$, we deduce from the convergence of $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p$ and Mertens' formula that $\sigma_N \sim c \log N$ for some $c \neq 0$.
This yields $\displaystyle\sum_{n \leqslant N} \mathrm{e}\big(a \Omega_{\mathcal{P}}(n)/b\big) \gg N$ as required. $\square$

Some open questions

Obtain a lower bound for the simple discrepancy of $\xi_{b,\mathcal{P}}$.

Can we go beyond frequency of digits in $\xi_{b,\mathcal{P}}$ perhaps with weighted distribution?

How to combine Lebesgue measure 0 properties and obtain normality?

Develop the theory of Poisson generic real numbers.

Poisson generic numbers

Fix integer $b \geqslant 2$. Let $Z^{\lambda}_{j,k}(x)$ be the proportion of words of length k that occur exactly j times in the first $\lfloor \lambda b^k \rfloor$ digits of the base-b expansion of x,

$$Z^{\lambda}_{j,k}(x) = \frac{\#\{w \in \{0,..,b-1\}^k : |x[1:\lambda b^k + k)|_w = j\}}{b^k}$$

Example for $b=2$, $\lambda=1$, $k=3$, $\lambda b^k = 8$,

$x = 10000100\ldots$

For $j=0$, $Z^{\lambda}_{j,k}(x) = 4/8$ (witnesses $011, 101, 110, 111$)
For $j=1$, $Z^{\lambda}_{j,k}(x) = 2/8$ (witnesses $001, 010$)
For $j=2$, $Z^{\lambda}_{j,k}(x) = 2/8$ (witnesses $100, 000$)
For $j \geqslant 3$, $Z^{\lambda}_{j,k}(x) = 0$

Definition (Zeev Rudnick) Let integer $b \geqslant 2$ and let λ be a positive real number. A real x in the unit interval is λ-Poisson generic if for every non-negative integer i,

$$\lim_{k\to\infty} Z^{\lambda}_{j,k}(x) = e^{-\lambda}\frac{\lambda^j}{j!}.$$

A real is Poisson generic if it is λ-Poisson generic, for all positive λ.

C. Aistleitner, V. Becher and O. Carton. "Normal numbers with digit dependencies", *Transactions of American Mathematical Society*, 372(6): 4425–4446, 2019.

V. Becher, A.Marchionna and G. Tenenbaum. "Simply normal numbers with digit dependencies", *Mathematika*, 69(4):988-991, 2023.

V. Becher, O. Carton, and P. A. Heiber. Finite-state independence. *Theory of Computing Systems*, 62(7):1555–1572, 2018.

N.Álvarez, V.Becher, E. Cesaratto, M.Mereb, Y. Peres and B. Weiss. "Poisson genericity in numeration systems with exponentially mixing probabilities", submitted, 2025.

N.Álvarez, V.Becher and M.Mereb. "Poisson generic sequences", *International Mathematics Research Notices* rnac234, 2022.

V. Becher and G. Sac Himelfarb. "A construction of a λ-Poisson generic sequence", *Mathematics of Computation* 92:1453-1466, 2023.

Y.Bugeaud. "Nombres de Liouville et nombres normaux". *Comptes Rendus de l'Académie des Sciences Paris*, 335(2):117–120, 2002.

J. Cassels. On a problem of Steinhaus about normal numbers. *Colloquium Mathematicum*, 7:95–101, 1959.

K. Jacobs and M. Keane. 0-1 sequences of Toeplitz type. *Z. Wahrsheinlichkeitstheorie verw. Geb.*, 13:123–131, 1969.

L. Liao and M. Rams. "Normal sequences with given limits of multiple ergodic averages. " *Publicacions Matemàtiques* 65(1) : 271–290, 2021.

W. Schmidt. Über die Normalität von Zahlen zu verschiedenen Basen. *Acta Arithmetica*, 7:299–309, 1961/1962.

G. Tenenbaum. "Introduction to analytic and probabilistic number theory", vol 163 Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, third edition, 2015.

G. Tenenbaum. "Moyennes effectives de fonctions multiplicatives complexes." *The Ramanujan Journal*, 44(3):641–701, 2017. Correction in: *The Ramanujan Journal* 53:1:243–244, 2020.

R. Tijdeman. " On integers with many small prime factors." *Compositio Mathematica*, 26(3):319–330, 1973.

Some ideas on the proofs

Theorem 1: Let $b \geqslant 2$, $\mathcal{P} = \{2\}$ and μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to all bases that are multiplicatively independent to b.

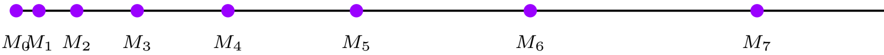
Proof idea

Let positive integer r be multiplicatively independent to b.
To prove μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$ are normal to base r, by Weyl's criterion, show that the sequence of fractional parts of $x, rx, r^2x, \ldots$ is u.d. in $[0, 1]$.
That is, we have to show that for every non-zero integer h,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e(r^j hx) = 0.$$

where,as usual, $e(x) = e^{2\pi ix}$.

Let M_k for $k = 1, 2, 3 \ldots$ be subexponential (such as $M_k = \sum_{j=1}^{k} \lceil e^{\sqrt{j}} \rceil$). so that for every N there is a k such that $N - M_k$ is $o(N)$.



Show for every k large enough,

$$\frac{1}{M_k - M_{k-1}} \sum_{M_{k-1} \leqslant j < M_k} e(r^j hx) < \varepsilon_k, \qquad \text{with } \varepsilon_k \to 0, \text{ as } k \to \infty$$

1 Fix $b, \mathcal{P}, r$. Fix a positive h. For each k define

$$Bad_k := \Big\{ x \in \mathcal{T}_{b,\mathcal{P}} : \frac{1}{M_k - M_{k-1}} \sum_{M_{k-1} \leqslant j < M_k} e(r^j h x) > 1/k. \Big\}$$

2 Show $\mu(Bad_k)$ is small enough to obtain a convergent series $\sum_k \mu(Bad_k)$.

By generalized Markov/Chebyshev inequality

$$\boxed{\mu(\{x \in X \ : \ |f(x)| > t\}) \quad < \quad 1/t^2 \int_X |f(x)|^2 \, d\mu(x)}$$

Our main lemma gives a suitable upper bound for

$$\int_0^1 \Big| \sum_{M_{k-1} \leqslant j < M_k} e(r^j h x) \Big|^2 d\mu(x)$$

3 Apply Borel Cantelli, obtain μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$ are outside $\bigcup_k Bad_k$.

4 For any N there is k such that $N - M_k = o(N)$.Then, μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} e(r^j h x) = 0.$$

5 Countably many h and r multiplicatively independent to b. $\square$

Proof of Theorem 1

Recall μ is the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$.

Lemma

Let $r \geqslant 2$ be multiplicatively independent to b.
Then for all integers $h \geqslant 1$ there exist constants $c > 0$ and $\ell_0 > 0$, depending only on b, r and h such that for all positive integers ℓ, m satisfying $\ell \geqslant m + 1 + 2\log_r b \geqslant m_0$,

$$\int_0^1 \left| \sum_{j=\ell+1}^{\ell+m} e(r^j hx) \right|^2 d\mu(x) \leqslant m^{2-c}.$$

Proof of Theorem 1

Using Euler's formula $e^{ix} = \cos x + i \sin x$, the Riesz-like product appears.

Lemma (adapted from Schmidt's Hilfssatz 5, 1961)

Let r and b be multiplicatively independent. There is $c > 0$, depending only on r and b, such that for all positive integers K and L with $L \geqslant b^K$,

$$\sum_{n=0}^{N-1} \prod_{\substack{k=K+1 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{b} + \frac{b-1}{b} \left| \cos \left(\pi r^n L b^{-k} \right) \right| \right) \leqslant 2N^{1-c}.$$

Lemma (Schmidt's Hilfssatz 5, 1961)

Let r and b be multiplicatively independent. There is $c_* > 0$, depending only on r and b, such that for all positive integers K and L with $L \geqslant b^K$,

$$\sum_{n=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n L b^{-k})| \leqslant 2N^{1-c_*}.$$

Schmidt's proof uses $|\cos(\pi x)|$ is periodic, $|\cos(\pi x)| \leqslant 1$ and $|\cos(\pi/b^2)| < 1$.

All these properties also hold for $\dfrac{1}{b} + \dfrac{b-1}{b} |\cos(\pi x)|$.

Theorem 2: Let $b \geqslant 2$, let $\mathcal{P}$ be a finite set of primes and let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to base b.

Proof idea

Let $\mathcal{P} = \{p_1, \ldots, p_r\}$ be a set of r primes.
Recall $\tau_{b,\mathcal{P}} : [0,1) \to \mathcal{T}_{b,\mathcal{P}}$ is defined as $\tau_{b,\mathcal{P}}(0.b_1 b_2 \ldots) = 0.a_1 a_2 \ldots$ where for each $n \geqslant 1$, $a_n = b_k$ where k is such that $n = j_k p_1^{e_1} \cdots p_r^{e_r}$ and j_k is not divisible by any $p_i \in \mathcal{P}$, $1 \leqslant i \leqslant r$.

The equivalence relation $\sim$ on the set of positive integers is defined as follows: $n \sim n'$ whenever there are exponents $e_1, \ldots e_r, e'_1, \ldots e'_r$ and a positive integer k such that

$n = j \cdot p_1^{e_1} \ldots p_r^{e_r}$,
$n' = j \cdot p_1^{e'_1} \ldots p_r^{e'_r}$,
j is not divisible by any $p \in \mathcal{P}$.

For example, for $\mathcal{P} = \{2, 3\}$, $2 \sim 3$, $3 \sim 36$, $36 \not\sim 5$.

Lemma (follows from Tijdeman 1973)

There is n_0 such that if $n' \sim n$ and $n' > n > n_0$, then $n' - n > 2\sqrt{n}$.

Proof of Theorem 2

The Toeplitz transform $\tau_{b,\mathcal{P}}$ induces a function $\delta : \mathbb{N} \mapsto \mathbb{N}$

$$\tau_{b,\mathcal{P}}(0.b_1 b_2 b_3 \cdots) = 0.a_1 a_2 a_3 \cdots = 0.b_{\delta(1)} b_{\delta(2)} b_{\delta(3)} \cdots$$

where $\delta(n) = \delta(n')$ exactly when $n \sim n'$. By the previous lemma,

$$\delta(n), \delta(n+1), \ldots, \delta(n + 2\lfloor\sqrt{n}\rfloor)$$

are pairwise different.

For each n, consider $a_n(x)$ as a random variable on space $([0,1), \mathcal{B}(0,1), \lambda)$. Since $a_n(x) = b_{\delta(n)}(x)$ for all n, two random variables a_n and $a_{n'}$ are independent, with respect to λ and μ, if and only if, $\delta(n) \neq \delta(n')$.

Thus, $b_{\delta(n)}, b_{\delta(n+1)}, \ldots, b_{\delta(n+2\lfloor\sqrt{n}\rfloor)}$ are mutually independent.

At every position n the number of independent variables is $2\lfloor\sqrt{n}\rfloor$ which exceeds the minimal required to ensure normality, established in Theorem 3.
$\square$

Theorem 4: We construct a number in $\mathcal{T}_{b,\mathcal{P}}$ for $b = 2$ and $\mathcal{P} = \{2\}$, normal to base 2.

We construct $x \in \{0,1\}^{\mathbb{N}}$ such that $x = even(x)$.
A word x is ℓ-perfect if each of the 2^ℓ many words of length ℓ occurs aligned in x the same number of times .

The construction consists in concatenating perfect sequences $s_1, s_2, \ldots$ such that $|s_{i+1}| = 2|s_i|$, $s_i = even(s_{i+1})$ and each s_i is ℓ_i-perfect for ℓ_i a power of 2.

Start with $s_1 := 01$, $s_2 := 1001$ and $\ell_2 := 1$. For $i > 2$,

If $|s_i| = \ell_i 2^{2\ell_i}$ and s_i is ℓ_i-perfect then construct s_{i+1} by transforming the n-th occurrence of u into $w = v \vee u$, where v is the n-th word of length ℓ_i in lexico order. Then s_i is $2\ell_i$-perfect, because all words of length $2\ell_i$ occur once in s_{i+1}. Set $\ell_{i+1} := 2\ell_i$.

If $|s_i| = m2^{2\ell_i}$, with m a multiple of ℓ_i but $m \neq \ell_i 2^{\ell_i}$, and s_i is ℓ_i-perfect then construct s_{i+1} as before, but now with multiplicity m. Notice that s_{i+1} is ℓ_i-perfect, each word of length ℓ_i occurs twice the number of times it occurred before. Set $\ell_{i+1} := \ell_i$.

$\square$