Constructing normal numbers

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Normal numbers

A base is an integer greater than or equal to 2.

For a real number x, the expansion of x in base b is a sequence $a_1 a_2 a_3 \ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

where infinitely many of the a_k are not equal to $b - 1$.

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A real number x is absolutely normal if x is normal to every base.

Normal numbers

Theorem (Borel 1922, Niven and Zuckerman 1951)

A real number x is normal to base b if, for every $k \geq 1$, every block of k digits occurs in the expansion of x in base b with limiting frequency $1/b^k$.

Not normal

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The numbers is the middle third Cantor set are not simply normal to base 3.

The rational numbers are not normal to any base.

Existence

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Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Constructions based on concatenation

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Let ( ) denote $\left(\frac{m}{2^x}, \frac{n}{2^x}\right)$; ( ) denote $\left(\frac{m'}{3^y}, \frac{n'}{3^y}\right)$; ( ) denote $\left(\frac{m''}{5^z}, \frac{n''}{5^z}\right)$.

0 **1**

Step t_0 ( )

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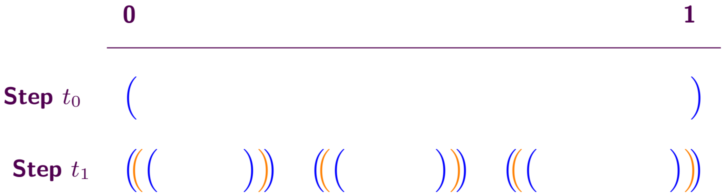

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Normal to all bases, non-effective constructions

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

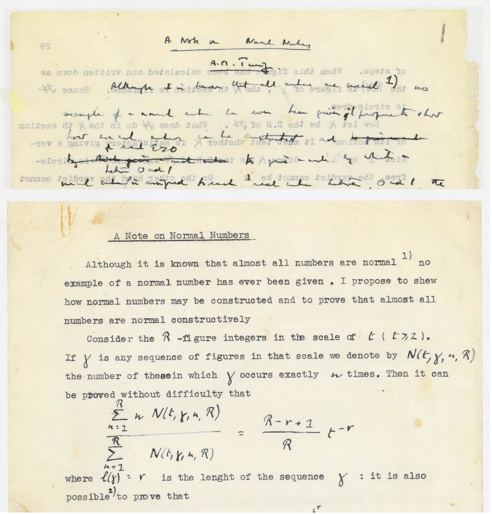
SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Normal to all bases, effective-construction

Alan Turing, A note on normal numbers. Collected Works, Pure Mathematics, J.L. Britton editor, 1992.



A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

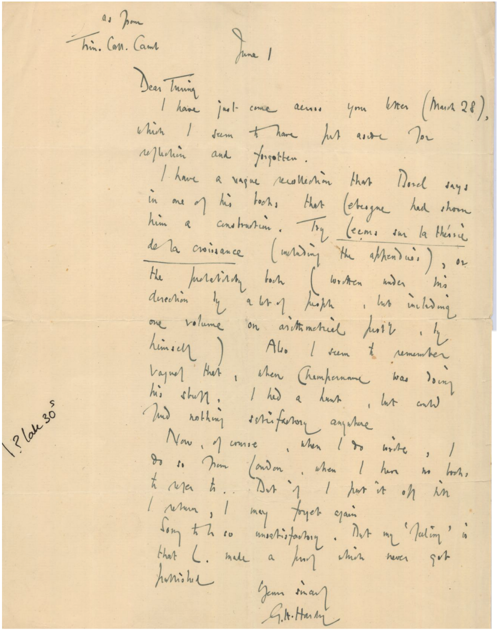
Consider the R -figure integers in the scale of t ($t \geqslant 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $l(\gamma) = r$ is the lenght of the sequence γ : it is also possible[2] to prove that

Corrected and completed in Becher, Figueira and Picchi, 2007.

Letter exchange between Turing and Hardy (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob, by himself). Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till I return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

1? late 30s

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Turing gives the following construction. For each k, n,

- $E_{k,n}$ is a finite union of open intervals with rational endpoints.
- Measure of $E_{k,n}$ is equal to $1 - \frac{1}{k} + \frac{1}{k+n}$.
- $E_{k,n+1} \subset E_{k,n}$.

For each k, the set $\bigcap_n E_{k,n}$ has Lebesgue measure exactly $1 - \frac{1}{k}$ and consists entirely of absolutely normal numbers.

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At each step, divide the current interval in two halves,

Choose the half that includes normal numbers in large-enough measure.

The output $\alpha(k,\nu)$ is the trace of the left/right selection at each step.

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Schmidt 1961/1962, Levin 1971 (proved in Alvarez and Becher 2015), Becher and Figueira 2002 gave other algorithms with exponential complexity.

Algorithm in polynomial time

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The algorithm is based on Turing's. Speed is gained by

- testing the extension instead of the whole initial segment.
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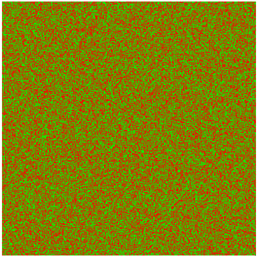
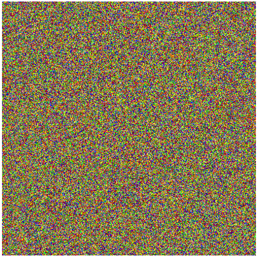
Lutz and Mayordomo (2013) and Figueira and Nies (2013) have another argument for an absolutely normal number in polynomial time, based on martingales.

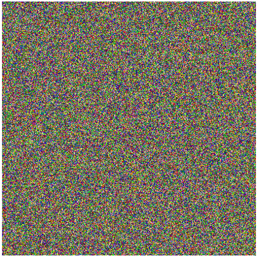
Algorithm in polynomial time

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

$0.40312905420038091323714283808270591027651167776241897751108966366\ldots$


base 2


base 6


base10

Plots of the first 250000 digits of the output of our algorithm.

Available from `http://www.dc.uba.ar/people/profesores/becher/software/ann.zip`

Open question

Is there an absolutely normal number computable in polynomial time having a nearly optimal rate of convergence to normality?

Constructions based on harmonic analysis

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ equidistributes modulo one for Lebesgue measure.

Normality and Weyl's criterion

Theorem (Weyl's criterion)

A sequence $(x_n)_{n\geq 1}$ of real numbers is uniformly distributed if, and only if, for every Riemann-integrable (complex-valued) 1-periodic function f, $\int_0^1 f(z)\ dz$ is the limit of the average values of f on the sequence.

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That is, if and only if, for every non-zero integer t, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i t x_k} = 0$.

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A number x normal to base b if and only if $(b^k x)_{k\geq 0}$ is u.d. modulo one.

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Multiplicative dependence

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Simple normality to base 8 implies simple normality to base 2 because $8 = 2^3$ and the digits in $\{0, .., 7\}$ correspond to the blocks in base 2:

$$000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \; 110 \quad 111$$

where half of the digits are 0.

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Theorem (Maxfield 1953)

Let b and b' multiplicatively dependent. For any real number x, x is normal to base b if and only if x is normal to base b'.

Normality to different bases

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Pollington 1981 showed the set of such numbers has full Hausdorff dimension.

Becher and Slaman 2014 refuted simple normality, a question of Brown, Moran and Pearce 1988.

Simple normality to different bases

Observation

If k is a multiple of ℓ, simple normality to b^k implies simple normality to b^ℓ.

Theorem (Long 1957)

Simple normality to infinitely many powers of b implies normality to base b.

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Necessary and sufficient conditions for a set S so that there exists a number that is simply normal to each of the bases in S and not simply normal to each of the bases in the complement of S.

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Moreover, the set of numbers with this condition has full Hausdorff dimension.

Also, the asserted real number is computable from the set S.

Normality to different bases in a logical perspective

Consider the Arithmetical Hierarchy of formulas in the language of first-order arithmetic.

Theorem (Becher and Slaman 2014)

Let S be a Π^0_3 set of bases closed by multiplicative dependence. There is a real x that is normal to every base in S and not normal to any of the bases in the complement of S. Furthermore, x is uniformly computable in the Π^0_3 formula defining S.

The proof shows that discrepancy functions are pairwise independent.

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Theorem (Becher and Slaman 2014)

The set of real numbers that are normal to at least one base is Σ^0_4*-complete.*

We conclude that the set of bases to which a number can be normal is not tied to any arithmetical properties other than multiplicative dependence.

Normal numbers and Diophantine approximations

Uniform distribution modulo one for appropriate measures

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Belief

If we consider appropriate measures, most elements of well structured sets are absolutely normal, unless the sets have evident obstacles.

Appropriate measures for normality

Let μ be a measure on the real numbers, The Fourier transform $\hat{\mu}$ of μ is

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{2\pi i t x} d\mu(x).$$

Lemma (direct application of Davenport, Erdős, LeVeque's Theorem 1963)

If μ is a measure on the real numbers such that $\hat{\mu}$ vanishes at infinity sufficiently quickly then almost every real number is absolutely normal.

Irrationality exponent

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs (p, q) with positive q.

Irrationality exponent

- Liouville numbers are the numbers with infinite irrationality exponent.
  Example: Liouville's constant $\sum_{n\geq 1} 10^{-n!}$.

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- Irrational algebraic numbers have irrationality exponent equal to 2.
  (Thue - Siegel - Roth theorem 1955).
- Rational numbers have irrationality exponent equal to 1.

Every real greater than or equal to 2 is the irrationality exponent of some real.

Becher, Bugeaud and Slaman (2015) considered the i.e. of computable numbers.

Absolute normality and irrationality exponents

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

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For every real a greater than or equal to 2, there is a real an absolutely normal number computable in a and with irrationality exponent equal to a.

Cantor-like fractals, measures and approximations

- Jarník (1929) and Besicovich (1934) defined a Cantor-like set for reals with a given irrationality exponent.
- Kaufman (1981) defined a measure on Jarník's set whose Fourier transform decays quickly.
- Bluhm (2000) refined it into a measure supported by the Liouville numbers, whose Fourier transform decays quickly.

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For the case of finite irrationality exponent, we considered the uniform measure on the fractal set given by the central halves of Jarník's intervals. Support is strictly included in support of Kaufman's measure and consists entirely of absolutely normal numbers.

Simple normality and irrationality exponents

Theorem (Becher, Bugeaud and Slaman, in progress)

Let S be a set of bases satisfying the conditions for simple normality.

- *There is a Liouville number x simply normal to exactly the bases in S.*
- *For every a greater than or equal to 2 there is a real x with irrationality exponent equal to a and simply normal to exactly the bases in S.*

Furthermore, x is computable from S and, for non- Liouville, also from a.

This theorem is the strongest possible generalization.

Open question

We would like several mathematical properties on top of normality.
Which sets admit an appropriate measure for normality?

Hochman and Shmerkin (2015) give a fractal-geometric condition for a measure on $[0, 1]$ to be supported on points that are normal to a given base. This support should have Lebesgue measure 1

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Based on concatenation of prescribed blocks

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Logarithmic complexity.

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Stoneham series (not in this talk)

1973	Normal to a given base.	Stoneham, Korobov
2012	Normal to base 2 but not to base 6	Bailey and Borwein

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These investigations on normal numbers aim to make progress in this direction.

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The End

Verónica Becher, Yann Bugeaud, Theodore Slaman. On simply normal numbers to different bases. Matematische Annalen in press, 2015.

Verónica Becher, Pablo Ariel Heiber, Theodore A. Slaman. A computable absolutely normal Liouville number. *Mathematics of Computation*, 232:1–9, 2015.

Verónica Becher, Theodore Slaman. On the normality of numbers to different bases. *Journal of the London Mathematical Society*, 90 (2): 472-494, 2014.

Verónica Becher, Yann Bugeaud, Theodore Slaman. Normal numbers and Diophantine approximations, in progress, 2015.

Verónica Becher, Yann Bugeaud, Theodore Slaman. The irrationality exponents and computable numbers, preprint 2014.

Verónica Becher, Pablo Ariel Heiber, Theodore A. Slaman. Normal numbers and the Borel hierarchy, *Fundamenta Mathematicae* 226: 63-77, 2014.

Verónica Becher, Pablo Ariel Heiber, Theodore Slaman. A polynomial-time algorithm for computing absolutely normal numbers. *Information and Computation*, 232:1–9, 2013.

Verónica Becher. Turings Note on Normal Numbers. In: *Alan Turing - His Work and Impact*. Ed. by S Barry Cooper and Jan van Leeuwen. Elsevier Science, 408–411, 2012.

Verónica Becher. Turings Normal Numbers: Towards Randomness. In: How the world computes - Turing Centenary Conference CiE 2012. Ed. by S.B. Cooper, A. Dawar, and B. Löwe. *Lecture Notes in Computer Science* 7318, 35–45 . Cambridge UK, 2012, .

Verónica Becher and Pablo Ariel Heiber. On extending de Bruijn sequences. In: *Information Processing Letters* 111.18, , 930–932, 2011.

Verónica Becher, Santiago Figueira and Rafael Picchi. Turing's unpublished algorithm for

Jarník's fractal

Fix a real a greater than 2. Jarník gave a Cantor-like construction of a set in $[0,1]$.
Let $(m_k)_{k\geq 1}$ be an appropriate increasing sequence of positive integers.
For each $k \geq 1$,

$$E(k) = \dot{\bigcup_{\substack{q \text{ prime} \\ m_k < q < 2m_k}}} \left\{ x \in \left(\frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \in \mathbb{N}, \left| \frac{p}{q} - x \right| < \frac{1}{q^a} \right\}$$

$E(k)$ has about $\dfrac{m_k^2}{\log m_k}$ disjoint intervals, each of length at least $\dfrac{2}{(2m_k)^a}$.

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Jarník's's fractal for the real a is

$$J = \bigcap_{k \geq 1} E(k).$$

Simple normality to different bases

The positive integers that are not perfect powers, $2, 3, 5, 6, 7, 10, 11, \ldots$ are pairwise multiplicatively independent. They are the minimal representatives of the equivalence classes of the multiplicative dependence relation.

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Theorem (Becher, Bugeaud and Slaman, 2015)

Let f be any function from the set of integers that are not perfect powers to sets of integers such that, for each b,

- *if for some k, b^k is in $f(b)$ then, for every ℓ that divides k, b^ℓ is in $f(b)$;*
- *if $f(b)$ is infinite then $f(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x simply normal to exactly the bases specified by f.

Moreover, the set of numbers with this condition has full Hausdorff dimension. Also, the real x is computable from the function f.