On clique-perfect graphs

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Università Tor Vergata, Febbraio/Marzo 2008

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- Perfect graphs
  - Basic definitions
  - Definition of perfect graphs
- Clique-perfect graphs
  - Definition
  - Relation with perfect graphs
- Clique graphs
  - K-perfect graphs
  - Hereditary K-perfect graphs
  - Clique subgraphs
  - Integer programming formulations
- Partial characterizations by forbidden induced subgraphs

Basic definitions

- A subgraph of a graph G is a graph such that its vertices and edges are vertices and edges of G, respectively.
- A subgraph of G induced by a subset of vertices of G is the subgraph containing those vertices and all the edges in G between them.
- The complement $\overline{G}$ of a graph G has the same vertex set but two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

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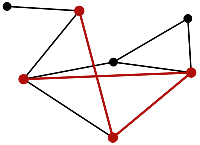

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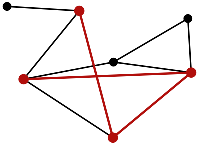

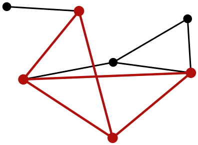

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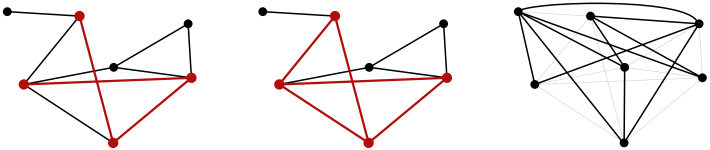

Intersection graphs

- Consider a finite family $\mathcal{F}$ of non-empty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.
- A circular-arc graph is the intersection graph of a finite family of arcs in a circle (such a family is called a circular-arc model of the graph).
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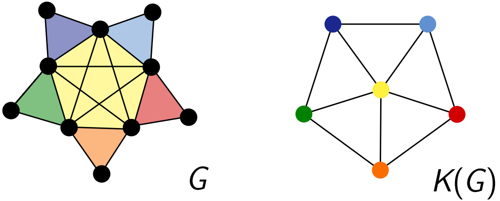

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- The line graph $L(G)$ of a graph G is the intersection graph of its edges.

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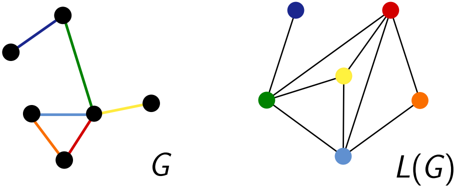

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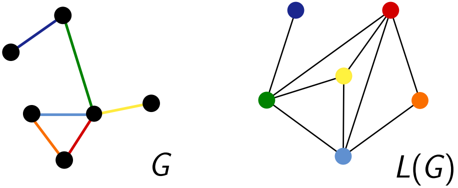

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The Helly property

- A family of sets $\mathcal{F}$ is said to satisfy the Helly property if every subfamily of $\mathcal{F}$, consisting of pairwise intersecting sets, has a common element.
- A graph is clique-Helly (CH) if its cliques satisfy the Helly property, and it is hereditary clique-Helly (HCH) if all its induced subgraphs are clique-Helly.

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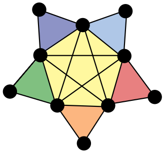

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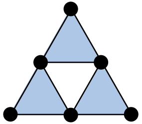

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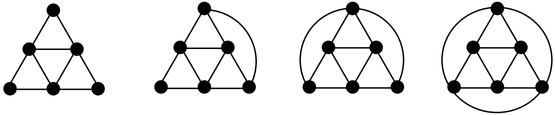
The Helly property

- Theorem (Prisner, 1993): A graph is hereditary clique-Helly iff it contains none of the following graphs as an induced subgraph.



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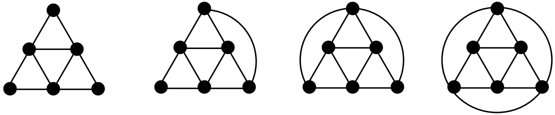

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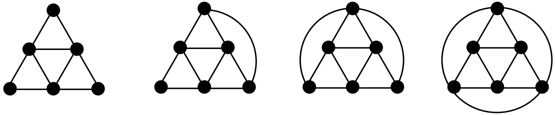

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The Helly property

- A graph is Helly circular-arc (HCA) if it admits a circular-arc model whose arcs satisfy the Helly property. Helly circular-arc graphs have polynomial time recognition (Gavril, 1974).

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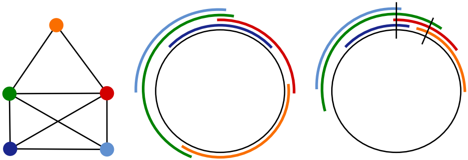
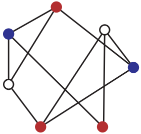


Qualche dubbio fino a qua ?

Chromatic number and maximum clique

- Coloring a graph consists of assigning colors to its vertices in such a way that no two adjacent vertices are given the same color.
- The minimum number of different colors needed to color a graph G is called the chromatic number of G and is denoted by $\chi(G)$.



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- In a coloring of G, the vertices of G having the same color must be pairwise non-adjacent. A set of pairwise non-adjacent vertices is called a stable set.

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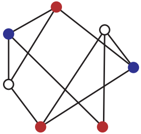

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Chromatic number and maximum clique

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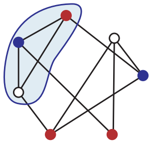

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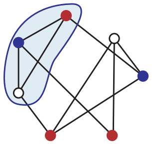

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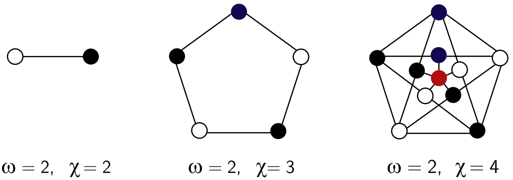
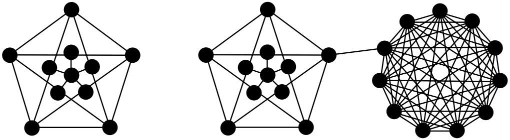
Mycielski's graphs

In 1955, Mycielski defined a family of graphs $\{G_k\}_{k\geq 0}$ such that $\omega(G_k) = 2$ and $\chi(G_k) = 2 + k$.



Mycielski's graphs

But, on the other hand, adding to any graph G a large enough clique (for example, with $|V(G)|$ vertices), it can be achieved a graph G' such that $\chi(G') = \omega(G')$.



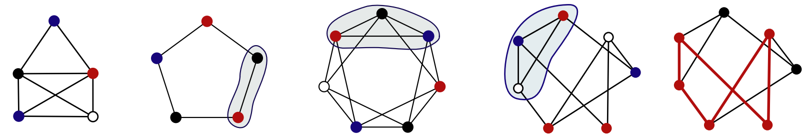
$\omega = 2$, $\chi = 4$ $\omega = 11$, $\chi = 11$

So, the equality of the parameters on the graph does not say much about the structure of the graph itself.

Perfect graphs

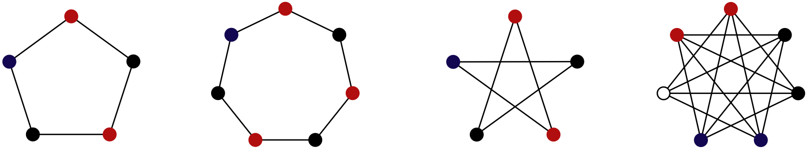
Berge defined perfect graphs in 1961. A graph G as perfect when $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Examples:



Holes and antiholes

- A hole C_n is a chordless cycle of length $n \geq 4$.
- An antihole is the complement of a hole.
- A hole or antihole is odd if it has an odd number of vertices (if n is odd).
- Odd holes and odd antiholes are not perfect:
  - $\chi(C_{2k+1}) = 3$ and $\omega(C_{2k+1}) = 2$
  - $\chi(\overline{C_{2k+1}}) = k+1$ and $\omega(\overline{C_{2k+1}}) = k$



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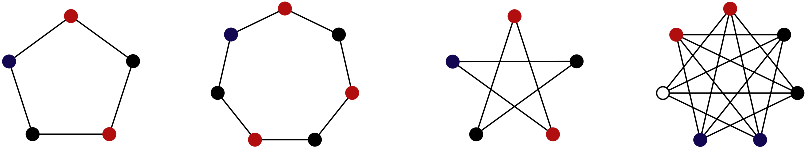

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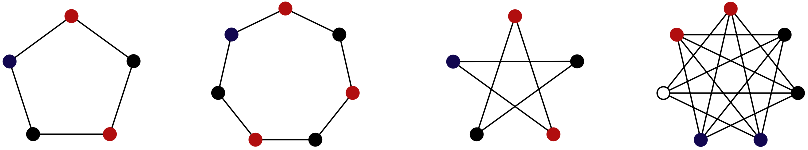

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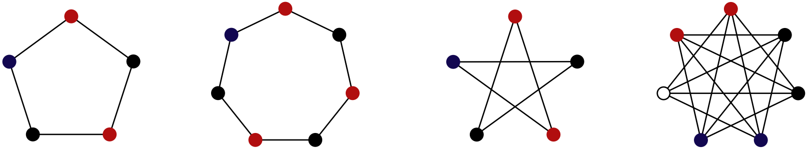

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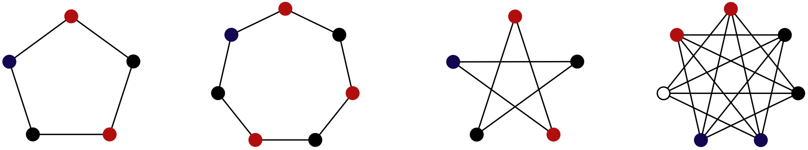

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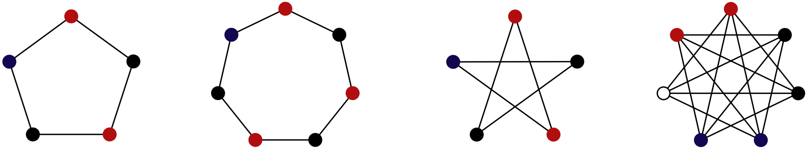


Classes of perfect graphs

- Bipartite graphs: graphs with chromatic number at most 2 (by definition) and their complements (Kőnig 1931).
- Line graphs of bipartite graphs and their complements (Kőnig 1916 and 1931, resp.).
- Comparability graphs and their complements (Dilworth 1950).
- Chordal graphs: graphs with no holes (Berge 1960) and their complements (Hajnal & Suranyi 1958).

Berge conjectures, now theorems

Berge conjectured in 1961 that a graph is perfect iff its complement is and, moreover, the only minimal imperfect graphs are odd holes and their complements.

Perfect Graph Theorem (Lóvasz–Fulkerson, 1972)

A graph is perfect iff it its complement is.

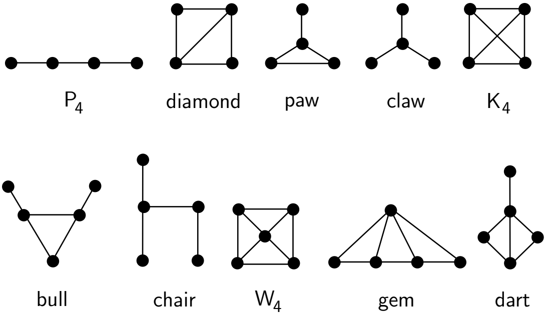
Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour and Thomas, 2002)

A graph is perfect iff it neither contains an odd hole nor an odd antihole as an induced subgraph.

Partial advances between 1961 and 2002

Asymptotical SPGT (Prömel & Steger 1992): Almost all Berge graphs are perfect.

Proof for graphs in special classes, some of them defined as F-free, where F is a graph.



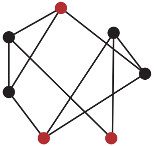
Partial advances between 1961 and 2002

- Verification of the SPGC for:
  - circle graphs (intersection graphs of chords of a circle) (Buckingham & Golumbic 1984)
  - planar graphs (Tucker 1973)
  - claw-free graphs (Parthasarathy & Ravindra 1976)
  - K_4-free graphs (Tucker 1984)
  - diamond-free graphs (Tucker 1987)
  - bull-free graphs (Chvátal & Sbihi 1987)
  - dart-free graphs (Sun 1991)
  - chair-free graphs (Sassano 1997)
  - square-free graphs (Conforti, Cornuéjols & Vušković 2001)

The proof for square-free graphs has an approach very close to that of the final proof.

Stable set and clique cover

- The stability number $\alpha(G)$ is the cardinality of a maximum stable set of G. It holds $\alpha(G) = \omega(\overline{G})$.
- A clique cover of a graph G is a subset of cliques covering all the vertices of G. The clique-covering number $\theta(G)$ is the cardinality of a minimum clique cover of G. It holds $\theta(G) = \chi(\overline{G})$. So, $\theta(G) \geq \alpha(G)$.
- By the PGT, a graph G is perfect if and only if $\alpha(H) = \theta(H)$ for every induced subgraph H of G.



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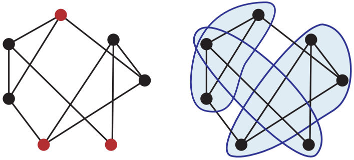

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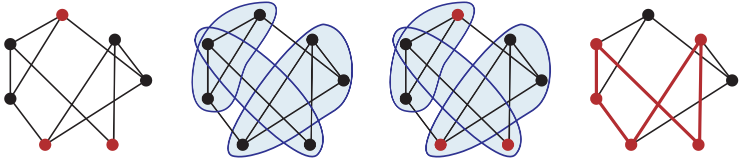

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Clique-independent sets and clique-transversals

- A clique-independent set is a collection of pairwise vertex-disjoint cliques. The clique-independence number $\alpha_c(G)$ is the size of a maximum clique-independent set of G.
- A clique-transversal of a graph G is a subset of vertices that meets all the cliques of G. The clique-transversal number $\tau_c(G)$ is the size of a minimum clique-transversal of G.
- Disjoint cliques must be covered with different vertices, so

$$\tau_c(G) \geq \alpha_c(G)$$

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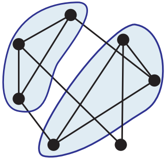

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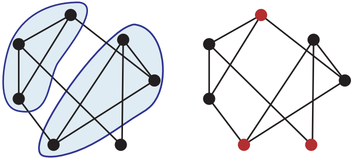

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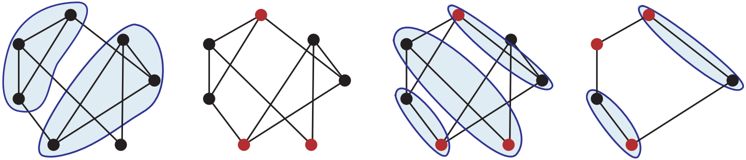
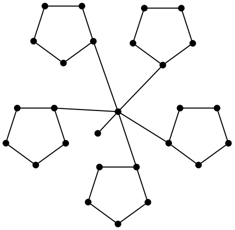


Highly clique-imperfect graphs

Durán, Lin and Szwarcfiter showed a family of graphs $\{G_k\}_{k\geq 2}$ such that $\alpha_c(G_k) = 1$ and $\tau_c(G_k) = k$ where number of vertices of G_k grows exponentially.

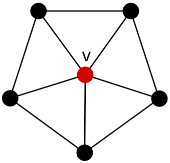
Later, Lakshmanan S. and Vijayakumar found another family of graphs $\{H_k\}_{k\geq 1}$ such that $\alpha_c(H_k) = 2k+1$ and $\tau_c(H_k) = 3k+1$ but H_k has only $5k+2$ vertices.



H_5

Clique-perfect graphs

However, adding a universal vertex v to any graph G (a vertex adjacent to every other vertex), it becomes $\alpha_c(G') = \tau_c(G') = 1$, since $\{v\}$ is a clique-transversal. So the equality in G does not give much information about the structure of the graph G.



A graph G is clique-perfect when $\alpha_c(H) = \tau_c(H)$ for every induced subgraph H of G.

Clique-perfect graphs

- The terminology “clique-perfect” has been introduced by Guruswami and Pandu Rangan in 2000, but the equality of the parameters α_c and τ_c was previously studied by Berge in the context of balanced hypergraphs.
- The complete list of minimal clique-imperfect graphs is still not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. (For perfect graphs there is a polynomial time recognition algorithm due to Chudnovsky, Cornuéjols, Liu, Seymour and Vušković).

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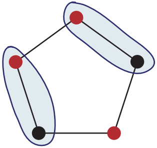
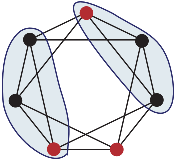
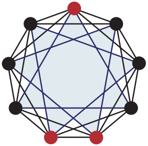
First question: is there some relation between clique-perfect graphs and perfect graphs?

- Odd holes $C_{2k+1}, k \geq 2$, are not clique-perfect: $\alpha_c(C_{2k+1}) = k$ and $\tau_c(C_{2k+1}) = k+1$.
- Antiholes $\overline{C_n}, n \geq 5$, are clique-perfect if and only if $n \equiv 0(3)$ (Reed, 2000): $\tau_c(\overline{C_n}) = 3$ and $\alpha_c(\overline{C_n}) = 2$ or 3, being 3 only if n is divisible by three.

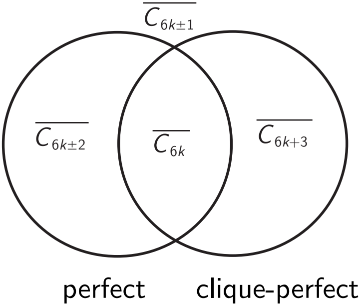

C_5


$\overline{C_7}$


$\overline{C_9}$

Relation with perfect graphs

So the classes overlap and we have the following scheme of relation between perfect graphs and clique-perfect graphs:



Triangle-free graphs

For triangle-free graphs (without isolated vertices) the cliques are the edges, so a maximum clique-independent set is a maximum matching, and a minimum clique-transversal is a minimum vertex cover.

Since odd holes are not clique-perfect, a clique-perfect triangle-free graph must be bipartite.

So, by Kőnig's theorem and analyzing the case with isolated vertices, every bipartite graph G satisfies $\tau_c(G) = \alpha_c(G)$. Moreover, since every induced subgraph of a bipartite graph is also bipartite, it follows that a triangle-free graph is clique-perfect iff it is bipartite.

Thus, within the class of triangle-free graphs, a graph is clique-perfect iff it is perfect.

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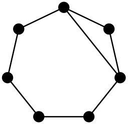
Families of clique-perfect graphs

- Complements of acyclic graphs.
- Comparability graphs (Balachandran, Nagavamsi & Pandu Rangan 1996).
- Dually chordal graphs (Brandstädt, Chepoi & Dragan 1997).
- $\{\text{gem}, W_4\}$-free graphs such that every odd cycle has a short chord.
- Distance-hereditary graphs (Lee & Chang 2006).
- Balanced graphs. A graph is balanced if its vertex-clique incidence matrix is balanced.


short chord

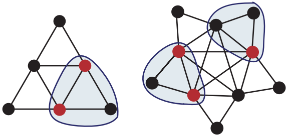
	v_1	v_2	v_3
M_1	1	1	0
M_2	0	1	1
M_3	1	0	1

Families of clique-imperfect graphs

- Odd holes.
- Antiholes of length not divisible by three.
- Odd suns.
- Odd generalized suns (they generalize odd suns and odd holes).
- Graphs S_k^1 and S_k^2, $k \geq 2$.
- Graphs Q_{6k+3}, $k \geq 0$.

Odd suns

An r-sun is a chordal graph with a cycle of length r and r vertices, each one of them is adjacent to the endpoints of an edge of the cycle.

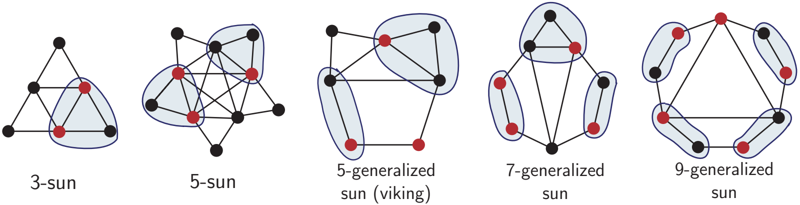


3-sun 5-sun

Odd suns are not clique-perfect: they have, as odd holes, $\alpha_c((2k+1)\text{-sun}) = k$ and $\tau_c((2k+1)\text{-sun}) = k+1$.

Odd generalized suns

A family of graphs generalizing both odd holes and odd suns are odd generalized suns. An edge in a cycle is non-proper if it forms a triangle with some vertex of the cycle. An odd generalized sun is formed by an odd cycle and a vertex for each non-proper edge, adjacent only to its endpoints.



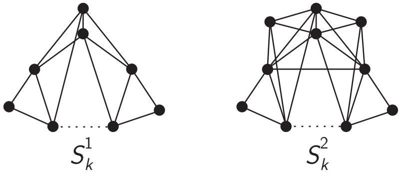
They have $\alpha_c((2k+1)\text{-gen. sun}) = k$ and $\tau_c((2k+1)\text{-gen. sun}) \geq k+1$.

Odd generalized suns

With the above definition, odd generalized suns are not necessarily minimal, and it is still an open question the characterization of minimal odd generalized suns and minimally clique-imperfect odd generalized suns.

Graphs S_k^1 and S_k^2, $k \geq 2$

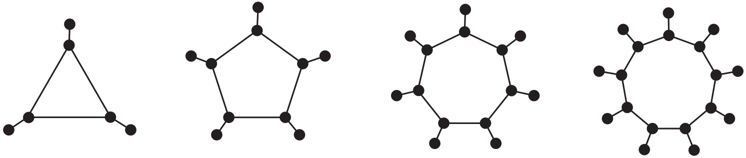
The families of graphs S_k^1 and S_k^2, $k \geq 2$, are defined based on a cycle of $2k+1$ vertices, as it can be seen in the figure, where dotted lines replace any odd induced path of length at least one.



They have $\alpha_c(S_k^i) = k$ and $\tau_c(S_k^i) = k+1$, for $i = 1, 2$.

Graphs Q_{6k+3}, $k \geq 0$

The family of graphs Q_n was defined by Szwarcfiter, Lucchesi and P. de Mello, 1998. For odd values of n, $\alpha_c(Q_n) = 1$ and $\tau_c(Q_n) = 2$ (they are exactly the graphs minimally clique-complete without a universal vertex).



$\overline{Q_3}$ $\overline{Q_5}$ $\overline{Q_7}$ $\overline{Q_9}$

But only the graphs Q_n with n odd and divisible by three are minimally clique-imperfect, the other ones contain clique-imperfect antiholes.

Partial advances on a forbidden induced subgraph characterization

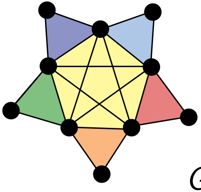
In the second part of the talk we will see partial characterizations, within the classes:

- Chordal graphs
- Diamond-free graphs
- Paw-free graphs
- P_4-sparse graphs
- $\{$gem,W_4,bull$\}$-free graphs
- Line graphs
- Complements of line graphs of bipartite graphs
- HCH claw-free graphs
- Helly circular-arc graphs

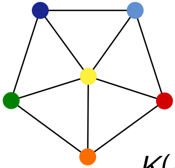
Qualche dubbio fino a qua ?

Parameters in the clique graph

- If we look at the clique graph, it holds the following relation:
  - $\alpha_c(G) = \alpha(K(G))$.
  - $\tau_c(G) \geq \theta(K(G))$, and, if G is clique-Helly, $\tau_c(G) = \theta(K(G))$.



G



$K(G)$

- So, in general $\alpha_c(G) = \alpha(K(G)) \leq \theta(K(G)) \leq \tau_c(G)$.

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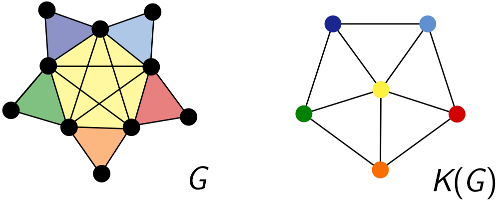

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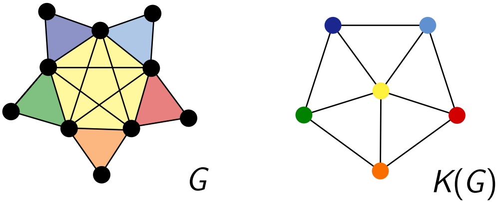

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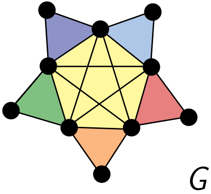

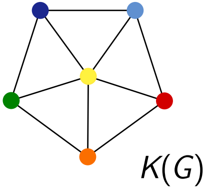

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K-perfect graphs

- A graph G is K-perfect when $K(G)$ is perfect.
- If a graph G is clique-Helly and K-perfect, then $\alpha_c(G) = \alpha(K(G)) = \theta(K(G)) = \tau_c(G)$.

Corollary

If a class of graphs is hereditary on induced subgraphs, clique-Helly and K-perfect, then the class is clique-perfect.

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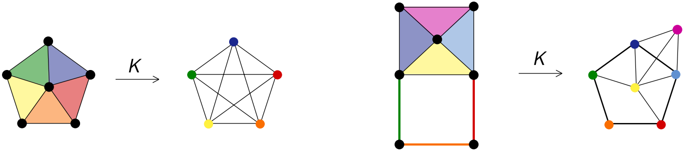
In general, it is **not** true that a clique-Helly graph is K-perfect iff it is clique-perfect.



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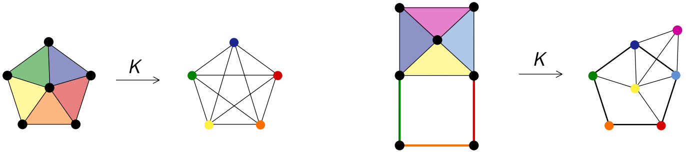

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Hereditary K-perfect graphs

The first item leads to defining hereditary K-perfect graphs as the graphs G such that every induced subgraph H of G is K-perfect.

Now, it holds:

Property

If G is a HCH and hereditary K-perfect graph, then G is clique-perfect.

The converse is not true.

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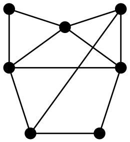

Hereditary K-perfect graphs

- For every $n \geq 4$, $K(C_n) = C_n$. So, a hereditary K-perfect graph cannot contain odd holes, since they are not K-perfect.
- On the other hand, it holds that $K(\overline{C_n})$ contains an induced C_5 for $n \geq 5$, n different from $6, 7, 9, 12$. But $\overline{C_6}$ is hereditary K-perfect, $K(\overline{C_7}) = \overline{C_7}$, and both $K(\overline{C_9})$ and $K(\overline{C_{12}})$ contain an induced $\overline{C_9}$.
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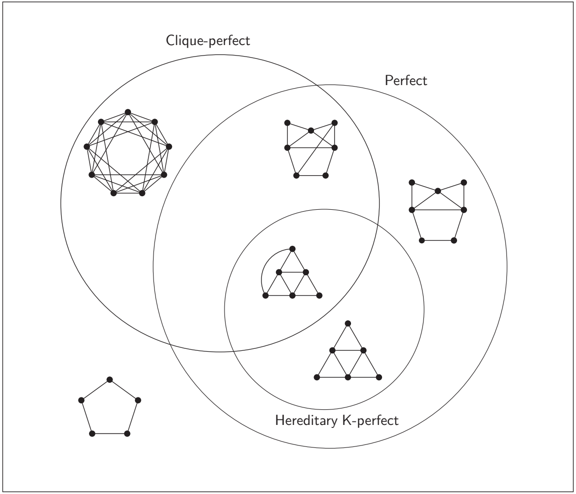
Relation between the three classes

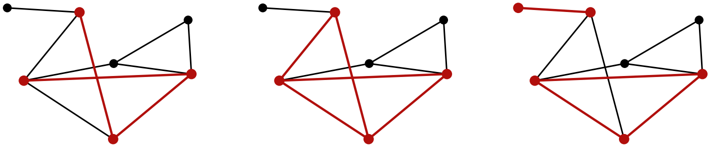


Clique subgraphs

The fact that "not every induced subgraph of $K(G)$ is the clique graph of an induced subgraph of G" leads to the definition of clique subgraphs.

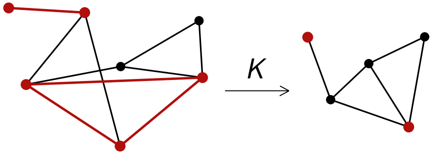
A subgraph H of G is a clique subgraph of G if all the cliques of H are also cliques of G.

Example: The second and third subgraphs are clique subgraphs (the third is not an induced subgraph).



Clique subgraphs

Now, we have that if H is a clique subgraph of G, then $K(H)$ is an induced subgraph of $K(G)$.



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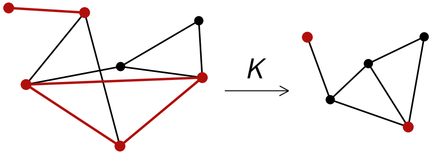
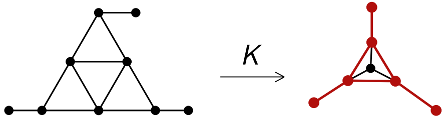


But, yet not every induced subgraph of $K(G)$ is the clique graph of a clique subgraph of G.



Clique subgraphs

Theorem (Prisner, 1993)

G is HCH iff for every family $M_1, \ldots, M_k$ of cliques of G, the subgraph of G formed by the vertices and edges of $M_1, \ldots, M_k$ is a clique subgraph of G and its cliques are exactly $M_1, \ldots, M_k$.

Corollary

If G is HCH, every induced subgraph of $K(G)$ is the clique graph of a clique subgraph of G.

c-Clique-perfect graphs

A graph G is c-clique-perfect if $\tau_c(H) = \alpha_c(H)$ for every clique subgraph H of G.

Theorem

If G is HCH, then G is K-perfect if and only if G is c-clique-perfect.

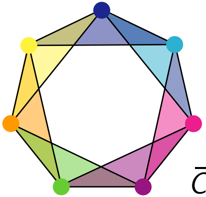
Proof. If G is clique-Helly, then every clique subgraph of G is clique-Helly.
$\Rightarrow$) Let H be a clique subgraph of G. Then $K(H)$ is an induced subgraph of $K(G)$. Since H is clique-Helly, $\tau_c(H) = \theta(K(H))$ and since $K(G)$ is perfect, $\tau_c(H) = \theta(K(H)) = \alpha(K(H)) = \alpha_c(H)$.
$\Leftarrow$) Let U be an induced subgraph of $K(G)$. Since G is HCH, let H be a clique subgraph of G such that $K(H) = U$. Since H is clique-Helly, $\tau_c(H) = \theta(U)$ and since G is c-clique-perfect, then $\theta(U) = \tau_c(H) = \alpha_c(H) = \alpha(U)$. □

Qualche dubbio fino a qua ?

Clique matrix

- The clique matrix A_G of a graph G has a row for each clique of G and a column for each vertex of G. $A_G(i,j) = 1$ if vertex j belongs to clique i and 0 otherwise.

Example:



$\overline{C}_7$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Derived graph and characterization of clique matrices

- We can regard a vector in $\{0,1\}^n$ as the characteristic vector of a subset of $\{1,\ldots,n\}$.
- So, we will say that two vectors $a, b \in \{0,1\}^n$ intersect if, for some $1 \leq i \leq n$, $a_i = b_i = 1$.

$$(1\ 0\ 0\ 1\ 1) \cap (0\ 1\ 0\ 1\ 1) = (0\ 0\ 0\ 1\ 1)$$

- We will say also that a vector a is included in another b if for every $1 \leq i \leq n$, $a_i \leq b_i$.

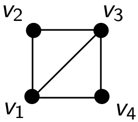
$$(1\ 0\ 0\ 1\ 0) \subseteq (1\ 0\ 1\ 1\ 0)$$

- Finally, a family of vectors will satisfy the Helly property iff their associated sets do.
- In this context, the derived graph G_A of a matrix A, is the intersection graph of its columns.

Derived graph and characterization of clique matrices

Example:

v_1	v_2	v_3	v_4
1	1	0	0
0	1	1	0
1	0	1	1



v_1	v_2	v_3	v_4
1	1	1	0
1	0	1	1

Note that $G_{A_G} = G$. The converse is not always true.

Theorem (Gilmore, 1960)

A 0-1 matrix A is the clique matrix of its derived graph iff:

1. A has no zero columns.
2. A has no included rows.
3. The columns of A satisfy the Helly property.

Integer programming formulations

Based on the clique matrix of the graph, $A_G \in \{0,1\}^{k \times n}$, the four parameters α, θ, α_c and τ_c can be formulated as integer programming problems.

$$\alpha(G) = \max\ 1 \cdot x \quad s.t. \quad A_G x \leq 1, \quad x \in \{0,1\}^n$$

$$\theta(G) = \min\ 1 \cdot y \quad s.t. \quad A_G^T y \geq 1, \quad y \in \{0,1\}^k$$

$$\tau_c(G) = \min\ 1 \cdot x \quad s.t. \quad A_G x \geq 1, \quad x \in \{0,1\}^n$$

$$\alpha_c(G) = \max\ 1 \cdot y \quad s.t. \quad A_G^T y \leq 1, \quad y \in \{0,1\}^k$$

Perfect matrices

A 0-1 matrix is perfect if the convex hull of the set $\{x \in \{0,1\}^n : \; A_G x \leq 1\}$ is the set $\{x \in \mathbb{R}^n : \; A_G x \leq 1, \; 0 \leq x \leq 1\}$.

Theorem (Chvátal 1975)

A 0-1 matrix with no zero columns nor included rows is perfect if and only if it is the clique matrix of a perfect graph.

Perfect matrices

- As a corollary, the maximum stable set problem of a perfect graph can be computed by linear programming.
- There is an algorithm of Tsukiyama, Idle, Ariyoshi and Shirakawa, that computes the clique matrix of a graph in $O(nmk)$, where n, m and k are the number of vertices, edges and cliques of the graph, respectively.
- But the difficulty is that a perfect graph can have exponentially many cliques, so it does not lead directly to a polynomial time algorithm for maximum stable set on perfect graphs.
- Nevertheless, Grötschel, Lovász and Schrijver proved in 1981 the existence of polynomial time algorithms for α, ω, θ and χ on perfect graphs.

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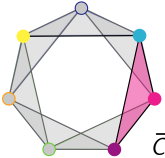
Clique matrices of subgraphs and the clique graph

- If H is an induced subgraph of G, then A_H can be obtained from A_G by selecting the columns corresponding to the vertices of H and then deleting the included rows.
- If H is a clique subgraph of G and G is HCH, then A_H can be obtained from A_G by selecting the rows corresponding to the cliques of H and then deleting the zero columns.
- If G is CH, then $A_{K(G)}$ can be obtained from A_G^T by deleting the included rows.

Example:



$\bar{C}_7$

1	1	0	0	0	0	1
1	1	1	0	0	0	0
0	1	1	1	0	0	0
0	0	1	1	1	0	0
0	0	0	1	1	1	0
0	0	0	0	1	1	1
1	0	0	0	0	1	1

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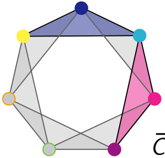
Example:



$\overline{C}_7$

1	1	0	0	0	0	1
1	1	1	0	0	0	0
0	1	1	1	0	0	0
0	0	1	1	1	0	0
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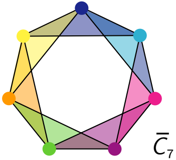
Example:



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Matrix characterization of clique-Helly K-perfect graphs

As a corollary of the last statement and Chvátal's theorem, we have the following:

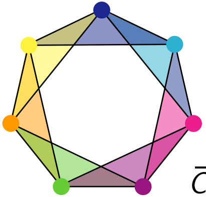
Corollary

A clique-Helly graph G is K-perfect if and only if A_G^T is a perfect matrix.

Balanced matrices

- A 0-1 matrix is balanced if it does not contain an odd square submatrix with exactly two 1's per row and per column.
- A graph is balanced iff its clique matrix is balanced.

Example:



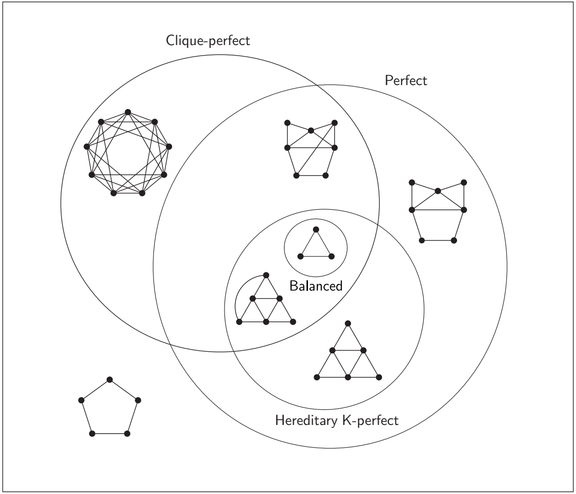
$\overline{C}_7$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Balanced graphs

- Balanced matrices are perfect matrices (Fulkerson, Hoffman & Oppenheim 1974), so balanced graphs are perfect graphs.
- Balanced graphs are HCH (Prisner 1993).
- Moreover, if A is balanced then A^T and every submatrix of A are balanced, so balanced graphs are K-perfect and they are an hereditary class.
- In conclusion, balanced graphs live in the intersection between perfect, clique-perfect and hereditary K-perfect graphs and, besides, they are HCH.

Relation between the classes



Balanced graphs

- For being HCH, balanced graphs have polynomially many cliques (Prisner 1993).
- Using the algorithm of Tsukiyama, Idle, Ariyoshi and Shirakawa, the clique matrix of a balanced graph can be computed in polynomial time. In particular, ω can be computed in polynomial time, and so $\chi = \omega$ since balanced graphs are perfect.
- Moreover, by the integrality of all the corresponding polyhedra, α, θ, α_c and τ_c can be computed in polynomial time for balanced graphs.
- Besides, balanced matrices (and so balanced graphs) can be recognized in polynomial time (Conforti, Cornuéjols & Rao 1999, and Zambelli 2005).

Qualche dubbio fino a qua ?

Partial characterizations by forbidden induced subgraphs

Chordal graphs

A graph is chordal when every cycle of length at least four has a chord. Chordal graphs have polynomial time recognition (Rose, Tarjan and Lueker, 1976).

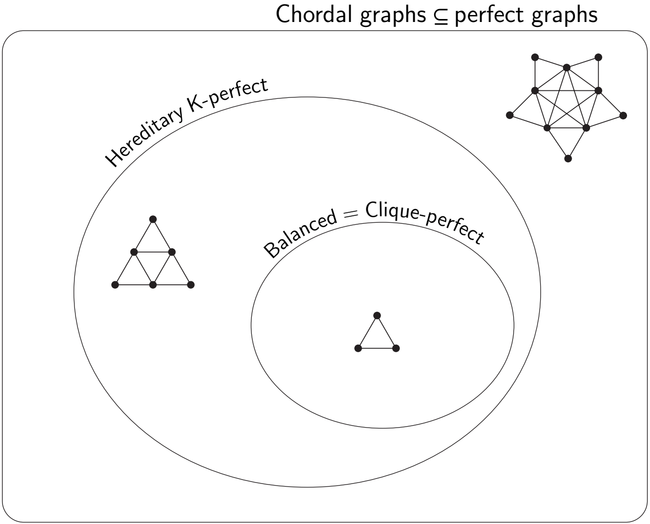
Theorem (Lehel and Tuza, 1986)

Let G be a chordal graph. Then the following are equivalent:

1. G does not contain odd suns.
2. G is balanced.
3. G is clique-perfect.

The recognition of clique-perfect chordal graphs can be reduced to the recognition of balanced graphs, which is solvable in polynomial time.

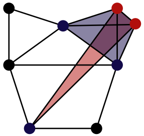
Chordal graphs



Operations preserving clique-perfectness

- Twin vertices: Two vertices v and w are twins in G if $N[v] = N[w]$, or, equivalently, if they belong to exactly the same cliques of G.

  Example:

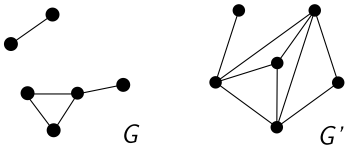


- If v and w are twins in G, then G is clique-perfect if and only if $G - v$ is. Moreover, $\alpha_c(G) = \alpha_c(G - v)$ and $\tau_c(G) = \tau_c(G - v)$.

Operations preserving clique-perfectness

- Disjoint union: Let $G = (V, E)$ and $G' = (V', E')$ with $V \cap V' = \emptyset$. Then the disjoint union $G \cup G'$ is the graph with vertex set $V \cup V'$ and edge set $E \cup E'$.

  Example:

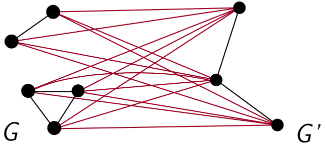


- Under these conditions, $G \cup G'$ is clique-perfect if and only if G and G' are. Moreover, $\alpha_c(G \cup G') = \alpha_c(G) + \alpha_c(G')$ and $\tau_c(G \cup G') = \tau_c(G) + \tau_c(G')$.
- Every graph is the disjoint union of its connected components.

Operations preserving clique-perfectness

- Join: Let $G = (V, E)$ and $G' = (V', E')$ with $V \cap V' = \emptyset$. Then the join $G \vee G'$ is the graph with vertex set $V \cup V'$ and edge set $E \cup E' \cup V \times V'$, that is, $\overline{G \vee G'} = \overline{G} \cup \overline{G'}$.

  Example:



- Under these conditions, $G \vee G'$ is clique-perfect if and only if G and G' are. Moreover, $\alpha_c(G \vee G') = \min\{\alpha_c(G), \alpha_c(G')\}$ and $\tau_c(G \vee G') = \min\{\tau_c(G), \tau_c(G')\}$.
- G is anticonnected if $\overline{G}$ is connected. Otherwise, it is the join of its anticomponents (the subgraphs of G induced by the vertices of the connected components of $\overline{G}$).

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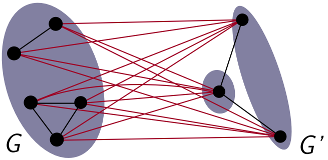

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Cographs

A cograph is a P_4-free graph, that is, a graph with no four vertices inducing P_4.

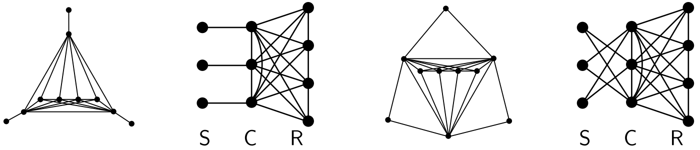
Theorem (Corneil, Lerchs and Stewart Burlingham, 1981)

Let G be a cograph. Then G is either trivial (it has only one vertex) or the disjoint union or the join of smaller cographs.

So, by induction and based on the properties stated previously about disjoint union and join, it can be proved that all the cographs are clique-perfect.

P_4-sparse graphs

A spider is a graph whose vertex set can be partitioned into three sets S, C and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a thin spider), or s_i is adjacent to c_j if and only if $i \neq j$ (a thick spider); R can be empty but if not, then R is complete to C and anticomplete to S.



Note that if G is a thin spider, then $\alpha_c(G) = \tau_c(G) = k$ (C is a clique-transversal an the legs are a clique-independent set).
Note also that a thick spider (that is not thin, so $k \geq 3$) contains an induced 3-sun.

P_4-sparse graphs

Def: Every set of five vertices contains at most one induced P_4.

Theorem (Hoàng, 1985)

Let G be a P_4-sparse graph. Then G satisfies one of this statements:

- G is trivial
- G is the disjoint union of smaller P_4-sparse graphs
- G is the join of smaller P_4-sparse graphs
- G is a spider (S, C, R), and R induces a P_4-sparse graph.

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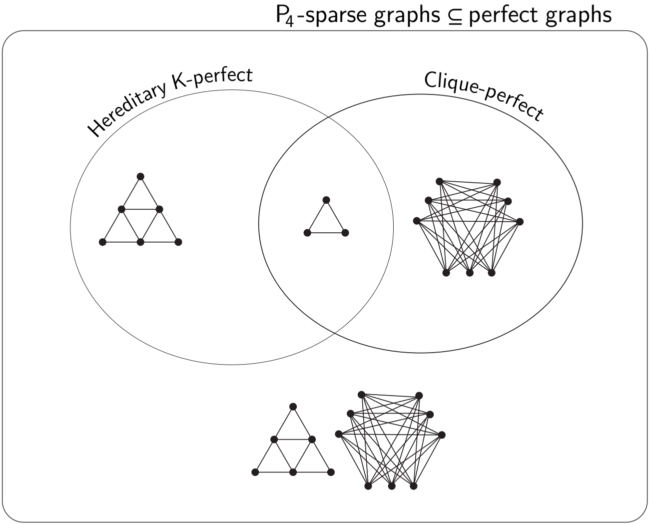
Theorem

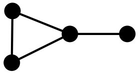
The only minimally clique-imperfect P_4-sparse graph is the 3-sun. So, if G is P_4-sparse, it is clique-perfect if and only if it does not contain 3-sun as an induced subgraph.

P_4-sparse graphs



Paw-free graphs

A paw-free graph is a graph with no four vertices inducing a paw.



Looking at the complement of a paw, it can be easily proved that if a paw-free graph is not anticonnected, then its anticomponents are stable sets.

It is also not difficult to prove that if a paw-free graph is connected and anticonnected, it contains no triangles. So, if it contains no odd-holes, then it is bipartite.

Then, the only minimally clique-imperfect paw-free graphs are the odd holes.

Paw-free graphs

We have then the following characterization, since odd antiholes of length at least 7 have induced paws.

Theorem

Let G be a paw-free graph. Then the following are equivalent:

1. G does not contain odd holes.
2. G is perfect.
3. G is clique-perfect.

Paw-free graphs

For the K-perfectness, every bipartite graph is K-perfect, because if G is bipartite, then $K(G) = L(G)$ and line graphs of bipartite graphs are perfect.

A non-anticonnected graph having an anticomponent of size less than three is also K-perfect:

- if G has an anticomponent of size one $\{v\}$, then v is a universal vertex, so $K(G)$ is complete.
- if G has an anticomponent of size two $\{v, w\}$, then every clique of G contains either v or w. The cliques containing v form a complete in $K(G)$, and so the cliques containing w. Therefore, $K(G)$ is the complement of a bipartite graph, thus perfect.

Paw-free graphs

It can be seen that the join of three stable sets of size three is not K-perfect. So, we obtain this characterization.

Theorem

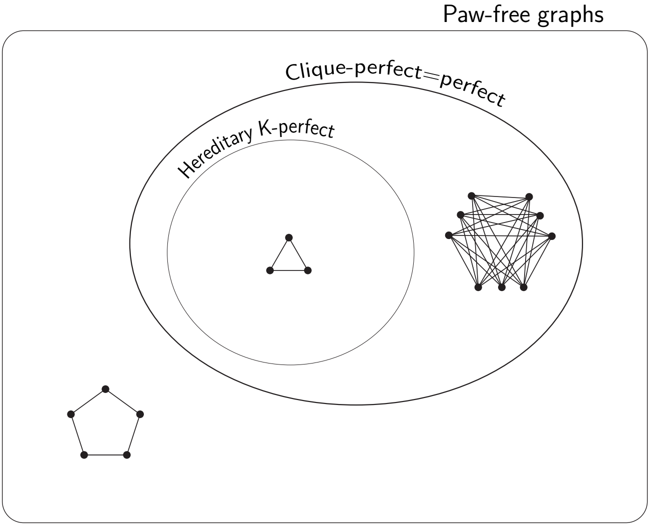
Let G be a paw-free graph. Then G is hereditary K-perfect iff each connected component H satisfies one of these conditions:

1. H is bipartite.
2. H is not anticonnected and at most two anticomponents of H have more than two vertices.

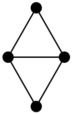
Theorem

The minimally K-imperfect paw-free graphs are odd holes and $\overline{3K_3}$.

Paw-free graphs



Diamond-free graphs



Theorem

Let G be a diamond-free graph. Then the following are equivalent:

1. G contains no odd generalized sun.
2. G is clique-perfect.
3. G is hereditary K-perfect.

Diamond-free odd generalized suns are odd generalized suns without non-proper edges. In this case, the characterization is not formulated by minimal subgraphs yet.

Sketch of proof

Theorem

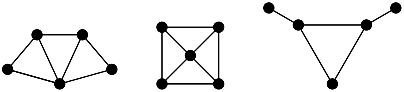
Let G be a diamond-free graph. Then the following are equivalent:

1. G contains no odd generalized sun.
2. G is clique-perfect.
3. G is hereditary K-perfect.

We prove $1 \Leftrightarrow 3$, and then the equivalence with 2 holds one way, because odd generalized suns are not clique perfect, and on the other way, by using that diamond-free graphs are hereditary clique-Helly.

Since K(diamond-free)=diamond-free, $K(G)$ cannot contain odd antiholes of length at least 7. Suppose $K(G)$ contains an odd hole. Consider the cliques $M_1, \dots, M_{2k+1}$ of G inducing that odd hole. Taking v_i in $M_i \cap M_{i+1}$ we have an odd cycle in G. It is easy to prove that if there is a non-proper edge in that cycle, then there is a diamond. So, that cycle induces an odd generalized sun with no improper edges.

{gem, W_4, bull}-free graphs



Theorem

Let G be a {gem, W_4, bull}-free graph. Then the following are equivalent:

1. G contains no odd holes.
2. G is perfect.
3. G is clique-perfect.
4. G is hereditary K-perfect.

In this case the proof is also based on the K-perfectness, but the arguments are more involved.

Line graphs

Let H be a graph. Its line graph $L(H)$ is the intersection graph of the edges of H. A graph G is a line graph if there exists a graph H such that $G = L(H)$. Line graphs have polynomial time recognition (Lehot, 1974).

Theorem

Let G be a line graph. Then the following are equivalent:

1. G contains no odd holes.
2. G is perfect.
3. G is hereditary K-perfect.

The proof is based on the structure of graphs whose line graph is perfect (several results by Trotter, de Werra and Maffray), and uses some of the operations preserving perfection or K-perfection.

Line graphs

For clique-perfection, we have this characterization.

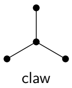
Theorem

Let G be a line graph. Then the following are equivalent:

1. no induced subgraph of G is and odd hole, or a 3-sun.
2. G is clique-perfect.

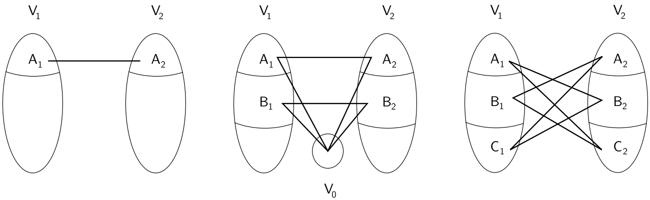
The proof is by induction, using as a base case when the graph is hereditary clique-Helly, because there we can use the K-perfection. Otherwise, we look how the other pyramids can appear and decompose the graph reducing the problem to smaller cases.

Claw-free hereditary clique-Helly graphs



Theorem (Chudnovsky and Seymour 2005)

Let G be a claw-free graph. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$, or G admits twins, or a non-dominating W-join, or a coherent W-join, or a 0-join, or a 1-join, or a generalized 2-join, or a hex-join, or G is antiprismatic.



Claw-free hereditary clique-Helly graphs

The characterization obtained for HCH claw-free graphs is the following:

Theorem

Let G be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of G is an odd hole, or $\overline{C_7}$.
2. G is clique-perfect.
3. G is perfect.
4. G is hereditary K-perfect.

Sketch of proof

Theorem

Let G be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of G is an odd hole, or $\overline{C_7}$.
2. G is clique-perfect.
3. G is perfect.
4. G is hereditary K-perfect.

The main part of the proof is $1 \Leftrightarrow 4$. The proof is by induction, based on the claw-free graphs decomposition theorem of Chudnovsky and Seymour. We prove it for the basic classes, and then we do induction using that if G in non-basic then it admits a decomposition. For some of the decompositions (1-join, 2-join) the idea is that these decompositions lead to some decompositions of the clique graph preserving perfection. Some other cases are more complicated, and require "brute force" to find the way of using induction.

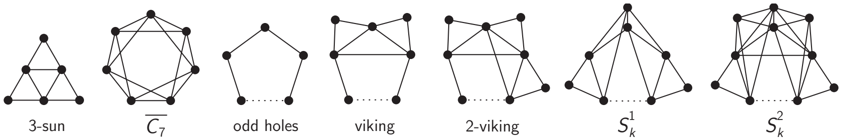
Helly circular-arc graphs

Recall that a graph G is HCA if there exists a family of arcs of a circle verifying the Helly property and such that G is the intersection graph of this family.

Theorem

Let G be a Helly circular-arc graph. Then the following are equivalent:

1. G does not contain any of the graphs in the figure, where the dotted lines replace an induced path of length at least one.
2. G is clique-perfect.



Sketch of proof

Theorem

Let G be a Helly circular-arc graph. Then the following are equivalent:

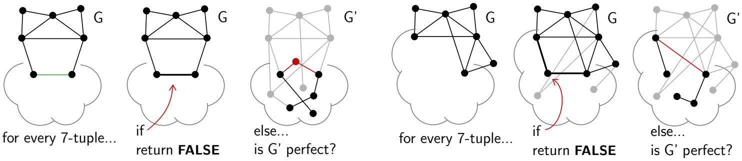
1. G does not contain any of the graphs in the figure, where the dotted lines replace an induced path of length at least one.
2. G is clique-perfect.

To prove $1 \Rightarrow 2$, we show that Helly circular-arc graphs which do not contain the graphs of the figure as induced subgraphs are K-perfect. This is the hardest part of the proof, and the idea is to "bring back" to G the odd holes and odd antiholes of $K(G)$. The remaining part is based in the fact that Helly circular-arc graphs that are not HCH have $\alpha_c = \tau_c$ or they are clique-complete without a universal vertex, and then we use a characterization of clique-complete graphs by Szwarcfiter, Lucchesi and P. de Mello, 1998.

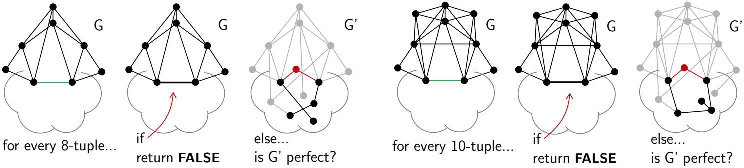
Recognition algorithm

Input: A *HCA* graph G; **Output:** **TRUE** if G is clique-perfect and **FALSE** if G is not.

1. Check if G contains a 3-sun. Case yes, return **FALSE**.
2. Check for odd holes and $\overline{C_7}$: check if G is perfect. Case not, return **FALSE**.
3. Check for vikings and 2-vikings:



4. Check for S_k^1 and S_k^2:



5. If no forbidden subgraph is found, return **TRUE**.

Summary

Class	Forbidden induced subgraphs
Chordal	odd suns
P_4-sparse	3-sun
Paw-free	odd holes
Diamond-free	odd generalized suns
$\{$gem, W_4, bull$\}$-free	odd holes
Line graphs	odd holes, 3-sun
HCH claw-free	odd holes, $\overline{C_7}$
HCA	3-sun, odd holes, $\overline{C_7}$, vikings, 2-vikings, S_k^1, S_k^2