An Asymmetric Multi-Item Auction with Quantity Discounts Applied to Internet Service Procurement in Buenos Aires Public Schools

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Abstract

This article studies a multi-item auction characterized by asymmetric bidders and quantity discounts. We report a practical application of this type of auction in the procurement of Internet services to the 709 public schools of Buenos Aires. The asymmetry in this application is due to firms' existing technology infrastructures, which affect their ability to provide the service in certain areas of the city. A single round first-price sealed-bid auction, it required each participating firm to bid a supply curve specifying a price on predetermined graduated quantity intervals and to identify the individual schools it would supply. The maximal intersections of the sets of schools each participant has bid on define regions we call competition units. A single unit price must be quoted for all schools supplied within the same quantity interval, so that firms cannot bid a high price where competition is weak and a lower one where it is strong. Quantity discounts are allowed so that the bids can reflect returns-to-scale of the suppliers and the auctioneer may benefit of awarding bundles of units instead of separate units. The winner determination problem in this auction poses a challenge to the auctioneer. We present an exponential formulation and a polynomial formulation for this problem, both based on integer linear programming. The polynomial formulation proves to find the optimal set of bids in a matter of seconds. Results of the real-world implementation are reported.

Keywords: multi-item auction, asymmetric bidders, quantity discounts, integer linear programming.

1 Introduction

Auction processes are methods of buying and selling goods and services that are potentially very efficient and thus can increase the economic welfare of all involved actors. For this to be the case, however, it is crucial that the design of the auction truly captures the actors' preferences and the nature of the goods or services. An auction mechanism can be defined as the specification of all possible bidding strategies available to the participants, and of an outcome function that maps these strategies to an allocation of items and corresponding payments the participants need to make or receive ([1]). The rise of the Internet and e-commerce have opened up a huge field of application for studying and improving auction mechanisms in which Operations Research is making a major contribution.

Real-world auctions frequently involve buying or selling multiple products. This leads to a *multi-unit* auction if all products are identical ([26]), or to a *multi-item* or *multi-object* auction if products differ from each other ([11]). On its hand, products are often composed of a complex mix of goods and services in which logistic factors play a central role. Since the material component of each unit of the product is identical by definition, each bidder's final unit price would supposedly be very similar. But due precisely to the logistic component, this simple criterion is not generally valid. Because product suppliers are heterogeneous, each will have cost advantages or disadvantages relative to its rivals in providing certain goods in the product mix. Also, a supplier with logistics facilities in a given area may find some increasing returns-to-scale in awarding several units in the auction. Such asymmetries among bidders create the conditions for what the literature refers as *asymmetric* auctions ([19], [30]).

An example of auctions with asymmetric bidders is the auction organized by JUNAEB, an agency of the government of Chile, for the contract to provide meals to the country's 5,000 public schools. Under the scheme some two million meals per day are served to school children 200 days a year at a cost to the public treasury of close to a billion dollars. The product delivered is a lunch composed of food items such as chicken or rice that are highly homogeneous goods whose quality can be specified in detail and are thus conceptually similar to a commodity. These food items must be transported, stored, cooked and served, thus posing a logistical problem of considerable complexity. In this case the spatial component is a key element in the logistics and therefore affects the quality of the product. There is a significant difference between supplying meals in Santiago, a major city of 6 million inhabitants, and in a rural area, leading firms to specialize and thus be efficient for certain supply conditions but not others. To optimize the purchase process a combinatorial auction ([9]) was designed in which the country is divided into territorial units that are the objects to be auctioned and can be grouped by the potential suppliers into packages they either accept or reject on an all-or-nothing basis. It is this characteristic that gives the school meals auction its combinatorial nature and allows each supplier to specify its full costs for each territorial unit more clearly. The process has operated successfully since 1997 ([6], [13], [12]).

Another example of auctions with asymmetric bidders is the procurement of bus routes in London ([15, 24]). The city's bus network consists of some 800 routes covering an area of 1,630 square kilometers and is used by more than 3.5 million passengers a day. Before the system was deregulated these services were run by a city-owned entity, but since privatization in 1984, provision has been awarded to transport firms through annual auctions. Implementation of the process was gradual, for although the first auction was held in 1985, it was not until 1995 that one-half of the network's routes had been auctioned at least once. Since then the system has stabilized, with 20% of the network auctioned each year. The arrangement is considered to be a success, having led to improved service quality and reduced costs for the public transport authority. The product supplied through the London bus auction can also be divided into two main components: the buses themselves, which are homogeneous, and the logistics involved in operating them, which include supplying fuel and lubricant, carrying out repairs, oil changes and other maintenance, and hiring and training professional bus crews. The logistic capabilities of the suppliers vary, each having different degrees of cost advantage or disadvantage relative to rival firms depending on the route. These variations arise due to asymmetries among the suppliers in such factors as the locations of their garages within the city, the locations of their crews' personal residences and their accumulated knowledge and skills in operating routes with different levels of demand.

Other examples of auctions with asymmetric bidders include highway procurement in California ([21]), snow removal contracts in Montréal ([14]) and electricity markets in Spain ([2]).

In this paper we examine the case of an asymmetric multi-item auction for supplying Internet services to public schools in the Argentinian capital of Buenos Aires. The asymmetries in this case arise because Internet service provision to any given school is highly constrained by the physical location of the existing supply infrastructure, the geographical distribution and capacity of each firm's installed technology. The case dates back to 2008 when the city government of Buenos Aires planned to invite tenders for provision of the Internet services to 709 schools locations over a two-year period. The original plan of the city contemplated an auction design in which participating firms bid separately on each school but were not required to bid on every one. The value of a bid would be the monthly price quoted by the bidder for service provision meeting certain previously specified technical requirements. For each school the winning bidder would be the firm submitting the best bid. This design had two main problems, however. First, no discounts for quantity were permitted, which had the effect of raising the average price bid by interested suppliers. Previous literature recognizes quantity discounts as an effective way to improve the outcome of auctioneers, for example, in the context of food and car manufacturers ([10, 16, 22]). Second, the asymmetries due to each firm's installed technology posed a limiting factor on its participation. This meant that in areas of the city with relatively little or no competition (i.e., few firms or a single firm with technology already installed), there was a risk that suppliers would submit high bids and even engage in collusion. Previous literature on auction theory recognizes competition as crucial for the profitability of an auction ([25]) which is also supported by empirical evidence ([33]). It was, therefore, a challenge for the city to procure the Internet service at fair prices in schools where there was no competition.

The present study reports on an auction proposal suggested by the authors that was eventually adopted by the city government authorities in place of their original plan. The proposed design is a single-round first-price sealed-bid multi-item auction, characterized by asymmetric bidders and quantity discounts. The aim of the alternative was to lower the city's costs. It allowed for the fact that not all participating firms were able to provide service to every school, and permitted them to offer quantity discounts. The schools were not considered as separate units; rather, the individual price the city paid to the supplier of each one depended on the number of schools that firm was awarded, and it is uniform for all the schools awarded to the same firm.

An auction process built around this format involves the development of mathematical models to determine which is the best set of bids for the auction organizer, and the implementation of algorithms to solve the models. In these tasks, Operations Research play a fundamental role. We formulate two integer linear programming (ILP) models aimed at minimizing the total cost of procurement. The first formulation, although correctly captures the combinatorial nature of the auction, has an exponential number of variables which can be permuted without changing the structure of the problem. In consequence, the formulation falls into a well-known phenomenon called *symmetry* ([29]), which makes it extremely difficult to solve. The second formulation has a polynomial number of variables and restrictions in the number of schools and bidders. The main family of variables in this formulation expresses how many schools are assigned to each winning bidder in each region or "competition unit" generated after the bids are submitted. This formulation, a significant methodological contribution of this paper, is more efficient than the first one in that it sidesteps the symmetry issues and was the one finally used in practice.

The rest of this article is organized in five sections. Section 2 reviews the various alternatives that were considered for the design of the school Internet service auction and describes the format finally settled on, which fully incorporates the possibility of quantity discounts and the bidding firms logistical constraints. Section 3 sets out two mathematical models for solving the problem of awarding the schools to bidders in such a way as to minimize total cost under the chosen format. Section 4 explains how all the optima of a given instance are obtained using an iterative process and proposes a simple algorithm for distributing the schools among the winning bidders in cases where there is more than one winner for a given competition unit. Section 5 describes the experience of applying the proposed auction, and finally, Section 6 presents our conclusions.

2 Design of the auction for Internet service procurement in Buenos Aires schools

In this section we set out various auction design alternatives we proposed and discuss the characteristics of each before describing the one that was finally implemented. As mentioned in the introduction, the original plan of the city contemplated an auction design in which participating firms bid separately on each school. The winning bidder for each school would be the firm submitting the best bid. The main objective at this stage was to develop a design that would generate competition between potential auction participants through quantity discounts in order to achieve the lowest price possible (the more schools allocated to a firm, the lower the unit price) while ensuring the process was transparent and not biased either towards or against any particular bidder. As noted above, the main factor in the bias issue is that Internet service provision to a specific physical site depends on the location of the technology infrastructure, and it is precisely this which defines the asymmetric nature of the auction. For a firm with no technology installed in the area of a given school, providing service to it would be very costly.

2.1 Auction based on territorial units

The city of Buenos Aires is divided into 21 school districts (see Figure 1). To include the ability to offer quantity discounts in the design we first considered the approach noted above for auctioning school meal services in Chile, with in this case the school districts as the territorial units. Firms would bid on combinations of districts and a mathematical model would allocate the award so as to achieve the most advantageous price structure. Each unit would be awarded entirely to a single firm, which would then have to provide Internet service to every school within that unit.

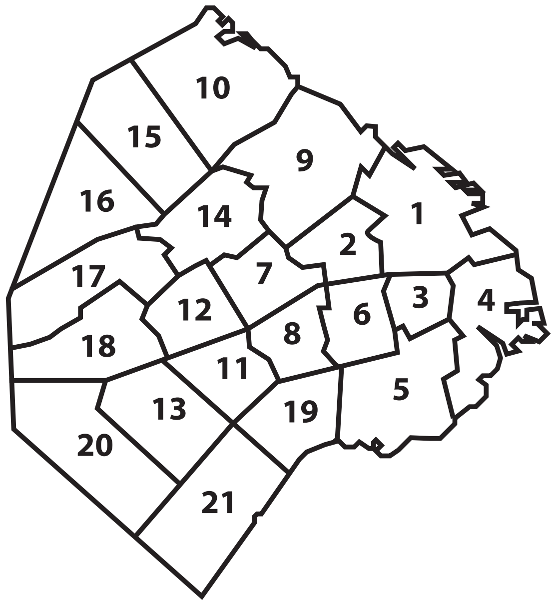

Figure 1: Buenos Aires school districts.

The advantage of this formulation was that it allows quantity discounts and therefore addresses the city's concern to achieve lower prices by promoting competition among potential suppliers. The disadvantage, however, was that it took no account of the firms' asymmetries due to their existing technology infrastructures. If the city school district division had been used, firms with technology installed in only certain parts of a district would have had to significantly raise their bids to finance an extension into the remaining parts. These providers could then have objected, and justifiably so, that the auction process was unfair. This particular territorial unit formulation had therefore to be rejected.

A different approach was then tried that began with an analysis of the radius of action of each potential bidder's installed technology. The radii so determined were superimposed on the school-district division to form a partition consisting of 11 territorial units, each aggregating various districts, that maximized the overlap between the units and the suppliers' existing coverage patterns (see Figure 2). But this alternative also had its shortcomings. For a start, the firms' radii of action were estimated by the city authorities rather than the firms themselves and were therefore subject to error; also, there was still a risk that some units would have little competition.

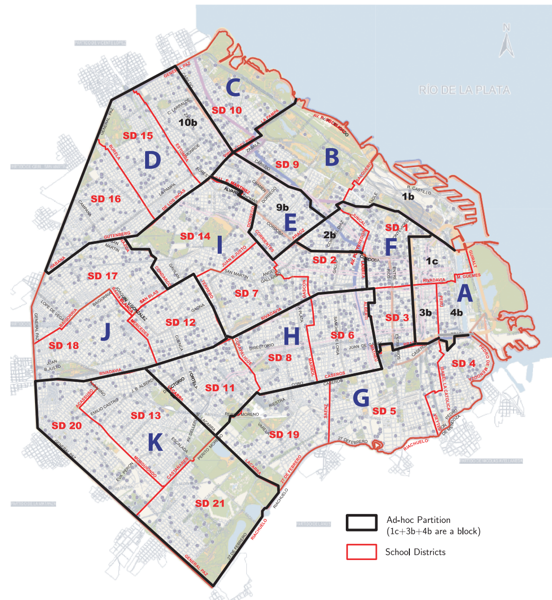

Figure 2: Division into territorial units based on firms' radii of action and school districts.

2.2 Multi-item auction based on competition units

In light of the above drawbacks, the idea of basing the auction on territorial units defined *a priori* was abandoned and efforts were redirected to the development of a method (the one eventually adopted) in which each firm submits a sealed bid indicating which schools it would supply and a schedule or curve of prices as a function of the quantity of schools it awards. The schedule must be non-increasing in the quantity of schools. For this purpose, the auctioneer informed the firms beforehand a set of quantity intervals T. Each interval t in T is delimited by a lower bound $\min_t$ and an upper bound $\max_t$. Thus, the bid submitted by firm i consists of a tuple $(S, c_1, \ldots, c_u)$, where S is the set of schools firm i is willing to supply, c_t is the price that firm i charges per school if it awards a quantity of schools in interval t, and u denotes the interval such that $|S| \in [\min_u, \max_u]$. Note the price in an interval is the same for all schools the firm bids on, and would thus be the "unit price per item" regardless of any given school's particular location. The city would then determine the set of winning bids in such a way as to minimize total cost, paying each chosen supplier the unit price specified in the schedule it submitted for the quantity it is to supply.

The resulting auction process is a single-round first-price sealed-bid auction with quantity discounts and which takes into account the asymmetries of the bidders by allowing them to specify the schools they are willing to supply given its logistical restrictions.

Note by specifying the set S on its bid a firm states that it is willing to provide the service in any subset of S. Thus, the auction can be interpreted as a combinatorial auction, where the packages of items that a firm bid on are all the subsets of schools contained in S.

Though for the case studied here a multi-round auction mechanism could have been used, it was rejected by the city authorities due to the complexity involved in its formulation (for a detailed analysis of multi-round auctions, see [3]).

Following previous literature ([6, 13]), it was suggested that an upper bound be placed on the number of schools that could be awarded to a single bidder to prevent the formation of monopolies, but the idea was rejected by the city, which saw no disadvantage in awarding all of the schools to the same firm if doing so meant getting the best price. Also, no cost had to be considered in the number of awarding firms in any given area of the city. It is important to remark here that, in addition to the schools, the market for internet suppliers include a number of other users, such as private householders and companies. These account for about 300,000 users, which provide conditions for the firms to keep competing even if only one or few of them would award all the schools.

A potential problem with the above mechanism (shared by the city's original proposal) is that a given school might only interest a single firm, which puts in a very high bid (though it would then be committed to the same high price for every other school it bids on). For this, the city sets a reserve price which is uniform for all schools. If for a given school there is no bid offering a price respecting this reserve price, the city could declare the auction void for that particular item and procure Internet service for it directly (for example, by contracting with the same firm privately at the going market unit price). Otherwise, however, this design is free of almost all of the problems that arise with the previous proposals and meets the objective of fostering competition, low costs and transparency.

- In terms of competition, since a single unit price within the same quantity interval must be quoted for all schools supplied, firms cannot bid a high price where competition is weak and a lower one where it is strong.
- In terms of cost, quantity discounts are captured through a quantity-graduated price schedule, resulting in lower total costs for the city.
- In terms of transparency, it takes into account the firms' existing technology infrastructures, by allowing firms to specify the sets of schools it is willing to provide the service. Thus, each firm can define its own radius of action based on its logistical restrictions.

The auction mechanism in this design is in fact a combination of (a) the city's original intention to auction each school separately (thus avoiding objections by potential bidders that there is too little overlap between the territorial units and their radius of action), and (b) the desire to increase competition by allowing bids to include quantity discounts. Since all schools are involved the mechanism is a multi-item auction, and since each bidder defines its own radius of action, the territorial units are still generated as regions of direct competition between bidders. A distinctive aspect is that these regions are defined subsequent to reception of the bids given that they are determined by the maximal intersections of the sets of schools each participant has bid on. The regions formed by these maximal intersections will also be referred to hereafter as *competition units*.

3 Mathematical formulation of the proposed auction

In this section we develop two integer linear programming models for optimally choosing the winning bids in the above-described multi-item auction by minimizing total cost to the city. The first model is an exponential formulation presented simply to demonstrate the combinatorial nature of the auction while the second one is an efficiently solvable model with a polynomial number of variables and constraints (on the number of schools and firms) and was therefore the specification actually used in practice. Both models search for the solution that awards all of the schools and is optimal for the city.

3.1 Exponential formulation

Let I be the set of firms (i.e., bidders) and E the set of schools. For each firm $i \in I$, we define $H_i \subseteq E$ as the set of schools the firm i bid on. For each $k = 0, \ldots, |H_i|$, we define as γ_{ik} the unit price per school bid by firm i if it is awarded exactly k schools. We also introduce the binary variable x_{iS} for each $i \in I$ and each subset $S \subseteq H_i$ such that $x_{iS} = 1$ if firm i is awarded exactly the set S of schools, otherwise $x_{iS} = 0$. With these definitions we can then state the integer linear programming model as follows:

$$\min \quad \sum_{i\in I}\sum_{S\subseteq H_i} \gamma_{i,|S|}\,|S|\,x_{iS} \tag{1}$$

$$\sum_{i\in I}\sum_{S\subseteq H_i:s\in S} x_{iS} = 1 \qquad \forall\ s\in E \tag{2}$$

$$\sum_{S\subseteq H_i} x_{iS} = 1 \qquad \forall\ i\in I \tag{3}$$

$$x_{iS} \in \{0,1\} \qquad \forall i\in I,\ \forall S\subseteq H_i \tag{4}$$

The objective function (1) attempts to minimize total cost. Constraints (2) specify that each school must be awarded to exactly one firm while constraints (3) impose that each firm must be awarded exactly one subset of the schools it bid on. The binary characteristic of the variables is stated in (4).

Note for the model to be feasible, all schools must be covered by at least one bid. Otherwise, the model would turn infeasible due to constraints (3). An alternative in this case would be to run the model by including in set E only the schools covered by at least one bid (this subset can be easily computed before building the integer programming model) and, for the remaining schools, the auctioneer would have to procure the service through another means. Other alternative would be to cancel the auction and perform it again encouraging providers to extend their bids. As we will report later, however, none of these alternatives were necessary in practice, as all schools were included in at least one bid.

Also, note since x_{iS} is defined for all $S \subseteq H_i$ and due to the quantity discounts, awarding a whole set to a given firm is at least as good for the auctioneer as awarding its separate parts to this firm. The subsets of H_i also includes the empty set, which has cost zero. Constraints (3) are helpful to identify the maximal set chosen for each company.

Model (1)-(4) explicitly reflects the combinatorial nature of the auction given that each winning bidder will in the end be awarded one subset of schools on which it bid at a unit price per school that depends on the number of schools in the award. However, since the number of variables in the model grows exponentially with the number of schools, the model is computationally impractical unless a column generation mechanism is implemented.

Furthermore, this formulation is highly symmetrical, because variable x_{iS} is defined for each $S \subseteq H_i$ and the cost contributed by firm i in the objective function is the same for subsets of the same cardinality. Note there are $\binom{H_i}{k}$ subsets of cardinality k that can be formed among the schools firm i bid on ($k \leq |H_i|$). This might turn to be a relatively high number. For example, if the bid of a firm contains 100 schools, there are more than 10^{13} different subsets containing 10 schools. Keeping a binary variable for each of these subsets, despite each of them would contribute the same cost in the objective function, might involve considerable computational effort. Moreover, if for a subset of schools $S \subseteq E$ various firms make bids and in the optimum S is partitioned among more than one firm, then all partitions of S that allocate the same number of schools to each firm are alternative solutions with the same objective function value. This property, widely studied in the literature (see, for example, [20, 29, 31, 32]), is known in ILP as *symmetry* and can increase the computational complexity of the solution enormously. This would be particularly problematic in the context of an auction process where, for the sake of fairness and transparency, all optimal solutions should be identified and submitted to the decision-makers. With so many equivalent solutions, finding all optimal solutions would be no simple task.

3.2 Polynomial formulation

This formulation is motivated by the fact that in the context of the present problem, for any given subset of schools on which the same firms are bidding, it suffices for optimization purposes just to determine *how many* schools from the subset are awarded to each firm; identifying *which* schools each firm is awarded can be determined a posteriori.

The schools are then partitioned into *regions* that are the classes of the equivalence relation defined such that two schools are equivalent if they have been bid on by exactly the same firms. Regions can be thought of as competition units whose schools are competed for by a given group of firms or just a single firm. The schools within each such unit are indistinguishable from one another in terms of the final award. The competition units are thus constructed after the bids are placed based on the schools the different subsets of firms bid on.

Recall E denotes the set of schools and I the set of firms. For simplicity, in set E we only consider schools that are included in the bid of at least one firm. We are given as an input the bids matrix $B \in \{0,1\}^{E \times I}$, such that $B_{ij} = 1$ if firm j bids for school i and 0 otherwise. We can then test if two schools are equivalent (with the equivalence definition above) in time $O(|I|)$. By using well known data structures and a naïve implementation we can find the set J of equivalence classes, i.e. regions, in time $O(|E|(|I| + \log |E|))$. For that and more involved algorithms see [17, 34].

If there is a region for which the set of firms bidding on it is empty, that region will not take part of the model and the auctioneer would have to attempt procuring the uncovered schools by other means.

Example 1. Suppose the set of schools is $E = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and the set of firms is $I = \{i_1, i_2, i_3\}$. Suppose the sets of schools covered by the bid of each firm are: $H_{i_1} = \{s_1, s_2, s_3, s_4, s_6\}$; $H_{i_2} = \{s_1, s_2, s_3\}$; $H_{i_3} = \{s_1, s_4, s_5, s_6\}$. Figure

3 illustrates this situation. Schools s_2 and s_3 are equivalent, and schools s_4 and s_6 are equivalent. We obtain $J = \{\{s_1\}, \{s_2, s_3\}, \{s_4, s_6\}, \{s_5\}\}$. The resulting competition units in this instance are: $\{s_1\}$, where all companies bid; $\{s_2, s_3\}$, where only firms i_1 and i_2 bid; $\{s_4, s_6\}$, where only firms i_1 and i_3 bid; and $\{s_5\}$, where only firm i_3 bids. These competition units are represented by the ovals in Figure 3, while the squares delimit the set of schools covered in the bid of each firm.

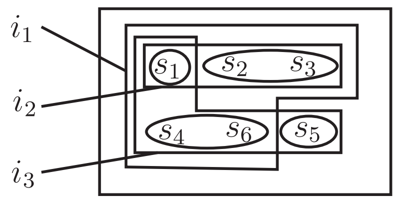

Figure 3: Illustration of an example with six schools, three firms and four competition units.

Once the competition units are computed, an integer linear model determines how many schools are awarded to each firm in each of the competition units generated by the bids. The formulation of this model is given below.

1. Model parameters:

    - I: set of firms;
    - J: set of competition units (regions), each one defined as the intersection of the schools contained in the bids by each subset of firms;
    - E_j: set of schools in competition unit $j \in J$ (recall by definition of competition units the sets E_j are disjoint);
    - I_j: set of firms bidding in competition unit $j \in J$ (the sets I_j are not necessarily disjoint);
    - J_i: set of competition units in which firm i has bid (the sets J_i are not necessarily disjoint);
    - T: quantity intervals (i.e., numbers of schools) making up the graduated price schedule. For this auction, the following intervals were arrived at in discussions with the city authorities: $T = \{0–19, 20–39, \ldots, 80–99, 100–149, 150–199, 200–299, \ldots, 600–699, 700–709\}$;
    - $\min_t$ and $\max_t$: the lower and upper bounds of quantity interval $t \in T$;
    - c_{it}: price per school in quantity interval $t \in T$ bid by firm $i \in I$ such that if a firm is awarded between $\min_t$ and $\max_t$ schools, it will charge a price equal to c_{it} for supplying each one.

2. Model variables:

    - $x_{ij} \in \mathbb{Z}_{\geq 0},\ j \in J,\ i \in I_j$: number of schools in competition unit j awarded to firm i;
    - $y_{it} \in \{0,1\},\ i \in I,\ t \in T$: variable defining whether quantity interval t is applied to firm i;
    - $z_{it} \in \mathbb{Z}_{\geq 0},\ i \in I,\ t \in T$: number of schools awarded to firm i in quantity interval t.

3. Formulation of the model:

$$\min \quad \sum_{i \in I} \sum_{t \in T} c_{it} z_{it} \tag{5}$$

$$\sum_{i \in I_j} x_{ij} = |E_j| \quad \forall\, j \in J \tag{6}$$

$$\sum_{j \in J_i} x_{ij} \geq \min{}_t - M(1 - y_{it}) \quad \forall\, i \in I,\ \forall\, t \in T \tag{7}$$

$$\sum_{j \in J_i} x_{ij} \leq \max{}_t + M(1 - y_{it}) \quad \forall\, i \in I,\ \forall\, t \in T \tag{8}$$

$$\sum_{t \in T} y_{it} = 1 \quad \forall\, i \in I \tag{9}$$

$$z_{it} \geq \sum_{j \in J_i} x_{ij} - M(1 - y_{it}) \quad \forall\, i \in I,\ \forall\, t \in T \tag{10}$$

$$x_{ij} \in \mathbb{Z}_{\geq 0} \quad \forall\, j \in J,\ \forall\, i \in I_j \tag{11}$$

$$y_{it} \in \{0,1\} \quad \forall\, i \in I,\ \forall\, t \in T \tag{12}$$

$$z_{it} \in \mathbb{Z}_{\geq 0} \quad \forall\, i \in I,\ \forall\, t \in T \tag{13}$$

The objective function (5) attempts to minimize total cost. Constraints (6) require that all schools in each competition unit must be covered. Constraints (7) and (8) link variables x and y such that $y_{it} = 1$ if the number of schools awarded to firm i falls within quantity interval t. M is the total number of schools in the auction, so in the present case $M = 709$. Constraints (9) impose that each firm must be associated with a single quantity interval. Constraints (10) force z_{it} to be at least the total number of schools assigned to firm i as long as $y_{it} = 1$, that is, as long as the firm is associated with quantity interval t. Constraints (11)–(13) specify the characteristics of the variables. The z variables can be defined as non-negative reals instead of non-negative integers (that is, $z_{it} \in \mathbb{R}_{\geq 0}$ for $i \in I$ and $t \in T$) given that in the optimal solution they will in any case be integers due to the model's constraints.

A noteworthy aspect of this model is that the number of variables and constraints is polynomial in the numbers of schools and firms given that the numbers of competition units and quantity intervals are bounded above by the number of schools. Furthermore, this formulation is more efficient than the more natural one (also polynomial in the number of schools) that would be obtained by considering each school individually, that is, with a binary variable for each school and each firm. Such a version, as well as containing more variables and constraints than the model just set out above, would have serious symmetry problems impacting negatively on the process of searching for multiple optima to be described in the next section.

4 Solution features

A solution to the polynomial formulation can be found by a commercial solver, as we will report in our results. Given an optimal solution, however, we still have to search for alternative optima and to allocate schools to the auction winners. These two tasks are addressed in the following subsections.

4.1 Search for alternative optima

It was noted earlier that when ILP models are applied to auctions, every optimal solution should be found and submitted to the auction organizer to ensure transparency and fairness between bidders. In cases where there is more than one optimal solution, the organizer will decide the final award based on whatever criteria they choose to apply.

To incorporate this consideration, once the optimum of the polynomial model is generated we add constraints to render this solution infeasible in all further iterations while maintaining every other solution feasible, and then run the model again. When a new optimum is found we check whether the objective function value is the same as or greater than the value for the previous optimum. This procedure is iterated as long as the same objective function value is obtained, thus generating all possible optimal alternatives. In what follows we describe the constraints that must be added at each iteration to exclude the optimum obtained in the immediately preceding one.

Let $x_{ij} = a_{ji}$ be the optimal solution for each firm $i \in I$ and each competition unit $j \in J$. For each value $a_{ji} > 0$ we add two binary variables, w_{ji} and w'_{ji}, which will take the value of 1 if $x_{ij} < a_{ji}$ and $x_{ij} > a_{ji}$, respectively, and 0 otherwise. To express these conditions and ensure that at least one of the x variables changes its value in each new optimal solution, the following constraints are added:

$$
\begin{aligned}
x_{ij} &\geq (a_{ji}+1)w_{ji} && \forall\ i \in I, j \in J \text{ such that } a_{ji} \neq 0 \\
M - x_{ij} &\geq (M-(a_{ji}-1))w'_{ji} && \forall\ i \in I, j \in J \text{ such that } a_{ji} \neq 0 \\
\sum_{a_{ji}\neq 0}(w_{ji}+w'_{ji}) &\geq 1
\end{aligned}
$$

Again, M is the number of schools. The addition of these new variables and constraints could potentially prolong the model's solution times, particularly if numerous iterations were performed to eliminate alternative optima. This could occur in particular if there were various firms with similar bids, giving rise to many different optimal solutions. Finally, it should be noted that this process of eliminating optimal alternatives is possible because the model simply determines quantities of "equivalent" schools to be assigned to each firm without specifying any further identifying details. Were it to do the latter, the number of optimal alternatives would be too high for the model to be practically solvable.

4.2 Distribution within a competition unit

Once the model has been solved and the final solution obtained, schools must be assigned to the winning bidders in each competition unit. As a first step, the schools in a competition unit where there is only one winning firm are assigned to the corresponding firm.

In a second step, the schools in competition units where there are two or more winning firms must be allocated to these firms. This step must be performed in a post-processing phase since the model does not assign individual schools to winning bidders, but only specifies the number of schools x_{ij} to be assigned to each winning bidder i in each competition unit j. Given the optimal value of the variables x_{ij}, there are multiple solutions to the problem of assigning schools to firms in a shared competition unit. This is not critical because the winning firms and their economical awards are already decided. Also, note that the chosen assignment does not change the value of the objective function identified by the optimal solution of the model.

In what follows we propose a simple procedure to find a unique assignment in cases where there is more than one winning bidder in a single competition unit. This procedure is a straightforward greedy algorithm that pans each region along a north-south or west-east axis, whichever is longer, and assigns first the required number of schools to the firm which in the first step awarded the northernmost (easternmost) school among the winning firms that share the competition unit. Suppose we must assign x_{ij} schools to this firm i in competition unit j. These schools are chosen as the x_{ij} northernmost (easternmost) schools of this competition unit. The procedure is then repeated iteratively with the remaining schools to assign the required numbers x_{ij} to each additional winning bidder i until all the schools of competition unit j have been assigned. By proceeding in this manner the "sub-regions" assigned to each winning bidder will tend to be compact areas, thus simplifying the logistics of providing the service. This algorithm is quite simple and easily implemented by a manual operator. Under any draw in the procedure (e.g., equal schools' coordinates or if no firm awarded a whole competition unit in the first step), we can choose randomly.

Example 2. Consider the situation illustrated in Figure 4. Schools $s_{(1,1)}$, $s_{(2,2)}$, $s_{(3,1)}$ and $s_{(4,2)}$ form a competition unit j_1 thoroughly awarded to firm i_1, who is the only bidder for these schools. Likewise, schools $s_{(7,2)}$ and $s_{(9,3)}$ form a competition unit j_2 thoroughly awarded to firm i_2. After that first step, the competition unit j_3 formed by schools $s_{(5,3)}$ and $s_{(6,1)}$ must be shared between i_1 and i_2, in such a way that $x_{i_1 j_3} = 1$ and $x_{i_2 j_3} = 1$. The north-south axis in this example is the longest one for the shared region. Therefore, firm i_1 gets awarded school $s_{(5,3)}$, which is the northernmost school of the competition unit j_3 we are splitting. Then, firm i_2 gets awarded the remaining school $s_{(6,1)}$.

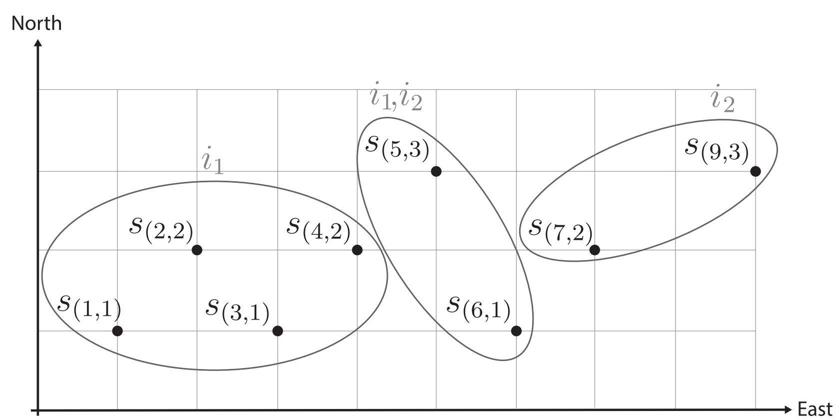

Figure 4: Illustration of an example with eight schools, two firms and three competition units.

More sophisticated procedures could be explored, but they would likely involve longer running times. For example, schools could be assigned in such a way as to minimize the maximum distance between any two assigned to the same bidder. Another possibility would be to use clustering algorithms with compactness requirements (see for example [23, 28]). Whichever procedure is used, it should ideally be disclosed by the auctioneer beforehand, for purposes of transparency and fairness of the auction. As reported in the following section, however, none of these procedures were necessary in the actual implementation.

5 Bids and results of the Internet procurement auction

Four firms participated in the auction process. Firm A bid on the entire set of 709 schools, which was foreseeable since the provider was known to already have coverage in the whole of Buenos Aires. Firm B bid on 348 schools covering the entire central zone of the city. Firm C bid on 99 schools in the city's northern zone and Firm D on 97 schools also on the north side, the wealthiest part of the city where greater competition was expected. The areas bid on by each firm are shown on the city map in Figure 5, and the six competition units determined once the bids were submitted are depicted in Figure 6. As can be seen, there were 248 schools in the southern zone of the city where Firm A was the only provider to enter a bid. Neither the city authorities nor Firm A itself knew that it would be the sole bidder in that area.

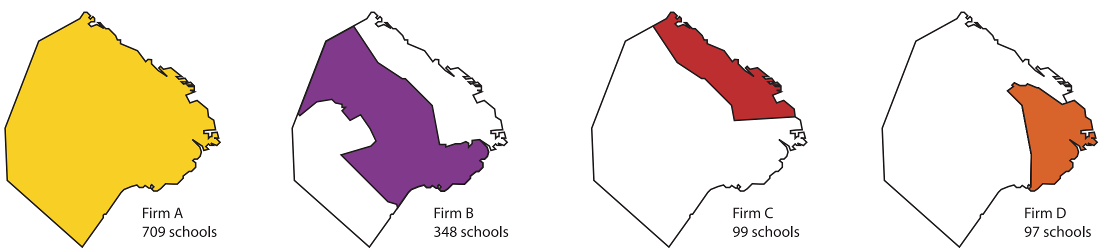

Figure 5: Areas of the city bid on by each firm.

The prices bid by each of the four firms are tabulated in Figure 7 for each quantity interval. The city authorities considered a good price to be about U$S 250 per month and the best (i.e., the most discounted) prices bid by all four firms were in that range. The bid prices are graphed in Figure 8. The schedule of prices of Firm A is flat in the first intervals and extremely steep at the end, which clearly reveals that this firm's bid was designed with the intention of supplying all 709 schools. This "all-or nothing" behavior is in line with previous literature ([7, 5]) which suggests the strongest or global bidder may behave strategically by favoring its bid for all items. While this behaviour could lead to inefficiencies in absence of synergies, it may also improve the outcome for the auctioneer when synergies are high.

The model was coded in Zimpl and solved using CPLEX solver package run-

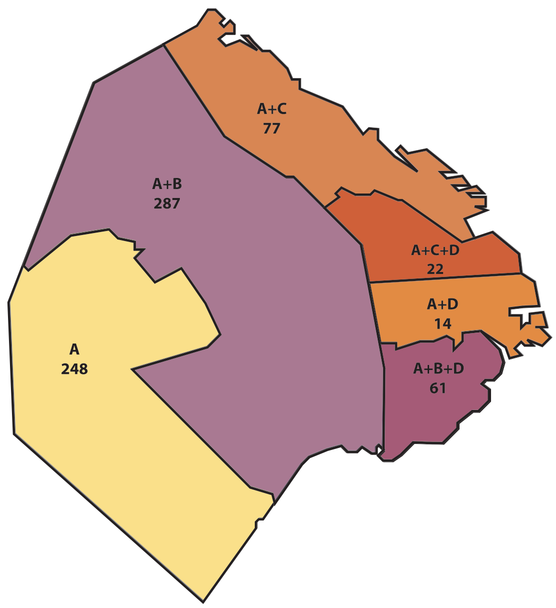

Figure 6: Competition units determined once bids were opened.

	Firm A		Firm B		Firm C		Firm D	
	Unit price: $ 1,174.18		Unit price: $ 665.50		Unit price: $ 497.92		Unit price: $ 422.51	
Interval	Discount	Unit price	Discount	Unit price	Discount	Unit price	Discount	Unit price
1 - 19	0%	$ 1,174.18	0%	$ 665.50	0%	$ 497.92	5%	$ 401.38
20 - 39	0%	$ 1,174.18	18%	$ 545.71	0%	$ 497.92	10%	$ 380.25
40 - 59	0%	$ 1,174.18	28%	$ 479.16	0%	$ 497.92	20%	$ 338.00
60 - 79	0%	$ 1,174.18	32%	$ 452.54	33%	$ 268.88	25%	$ 316.88
80 - 99	0%	$ 1,174.18	40%	$ 399.30	45.02%	$ 222.52	33%	$ 283.08
100 - 149	10%	$ 1,056.76	50%	$ 332.75	---	---	---	---
150 - 199	15%	$ 998.05	59%	$ 272.86	---	---	---	---
200 - 299	20%	$ 939.34	61%	$ 259.55	---	---	---	---
300 - 399	30%	$ 821.92	70.5%	$ 196.32	---	---	---	---
400 - 499	40%	$ 704.51	---	---	---	---	---	---
500 - 599	50%	$ 587.09	---	---	---	---	---	---
600 - 699	60%	$ 469.67	---	---	---	---	---	---
700 - 709	80%	$ 234.84	---	---	---	---	---	---

Figure 7: Bids by each firm in each quantity interval.

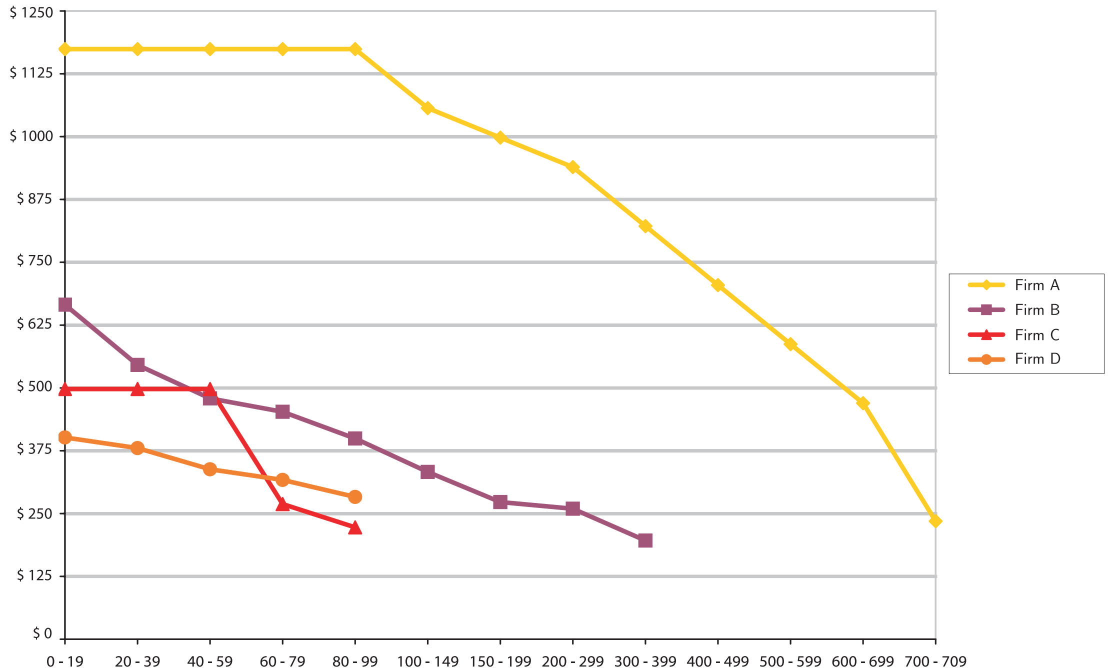

Figure 8: Graphical representation of bids by each firm in each quantity interval.

ning on a PC with a 1.6 GHz processor and 2 GB of RAM. Execution time was a matter of seconds. The model generated a single optimum solution of the problem that awarded all 709 schools to firm A at a total monthly cost to the city of U$S 166,501, implying a total expenditure for the two-year supply period of U$S 3,996,037. The average monthly unit cost was U$S 234.84, the price bid by the firm A for the last (most discounted) quantity interval.

6 Discussion and conclusions

This paper contributes to the literature on asymmetric multi-item auctions with a piece of empirical evidence from the procurement of Internet service to Buenos Aires public schools. The asymmetries among providers in this auction are due to the geographic location of their pre-existing infrastructure, which limit their ability to offer the Internet service to all schools. This affects the degree of competition within the different areas of the city. Since the details on the infrastructure of the firms are unknown for the auctioneer beforehand, it is a challenge to configure packages of schools to auction them together. At the same time, allowing the companies to bid on packages of schools is desirable so that they can reflect their economies of scales and the procurement cost for the auctioneer turns lower.

We proposed a design for this auction allowing for quantity discounts, which are specified in the bid of each firm by a schedule or curve of prices as a function of the number of schools it awards. The design introduced the concept of competition units, which are the maximal intersections of the sets of schools each participant has bid on. These units capture the fact that not all companies have infrastructure to provide the service in all areas of the city. The design prevents firms from bidding high prices in areas where competition is weak and low prices where it is strong by imposing that the unit price for all items bid on be identical.

The winner determination problem in the resulting auction is a challenge for the auctioneer. We presented two integer linear formulations for this problem. A first formulation, exponential in the number of schools, reflects the combinatorial nature of the auction but is computationally impractical. A second formulation, polynomial in the number of schools and firms, proved to be efficient and was the one used in practice.

Our work was implemented in the auction of a supply contract for Internet service to the 709 schools in the city of Buenos Aires. The polynomial formulation solved the problem of determining the best bid in just a few seconds. Based on the bids actually submitted in the Buenos Aires auction, the definitive solution awarded all of the schools to the same bidder (Firm A) at a unit price of U$S 234.84 per month. This is consistent with previous literature that predicts the strongest bidder is always the winner ([30, 8]).

Note the awarded price is 6% lower than the U$S250 regarded as good by the city beforehand. Using this price as reference and the auction format the city originally intended to perform (where all 709 schools were going to be auctioned separately), the savings generated by the multi-item format with quantity discount and the optimization model are estimated in about U$S 257,963 for the two-year supply period. Additional perspective on the effectiveness of the proposed auction format may be obtained by calculating what would have been the highest unit price Firm A could have bid for the most discounted quantity interval and still be awarded the contract for every school. This can be done by running the polynomial model presented in Section 3 with different values for Firm A's most discounted price. A binary search was done over the range of possible values for this price until the limit value was found. The highest price turned out to be U$S 401.38, or 71% above the actual bid value. This suggests that the auction achieved a competitive price in the whole city even if in some areas there was no competition for the strongest bidder. This adds some evidence in line with the experiments recently reported by [33], that advocate for a sealed-bid auction in the asymmetric multi-item auctions, because subjects in the role of strong buyers fail to take full advantage of their favorable value distribution. Empirical outcomes are also valuable in the debate on equilibrium of assymmetric auctions, for which most known results to date are derived numerically rather than analytically [19].

A direction for further research is to compare the design and solution approach proposed in this article with other designs and approaches, by means of an experimental study such as in [7]. Another direction is attempting to obtain a structural estimation or semiparametric estimation of this auction. Since some inputs are common to all firms but they differ based on the location of their infrastructure, a model within the affiliated private value paradigm which allows for dependence among bidders' private values is particularly appealing [27, 4, 18].

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