Partial Characterizations of Circle Graphs

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Abstract

A *circle graph* is the intersection graph of a family of chords on a circle. There is no known characterization of circle graphs by forbidden induced subgraphs that do not involve the notions of local equivalence or pivoting operations. We characterize circle graphs by a list of minimal forbidden induced subgraphs when the graph belongs to one of the following classes: linear domino graphs, P_4-tidy graphs, and tree-cographs. We also completely characterize by minimal forbidden induced subgraphs the class of *unit Helly circle graphs*, which are those circle graphs having a model whose chords have all the same length, are pairwise different, and satisfy the Helly property.

Key words: circle graphs, Helly circle graphs, linear domino graphs,

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1. Introduction

All graphs in this work are undirected, without multiple edges and without loops. Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ or co-G the complement of G. Let $X \subseteq V(G)$. The *subgraph induced* by X in G is denoted by $G[X]$. We define $G - X$ to be $G[V(G) \setminus X]$.

An *isolated vertex* is a vertex with no neighbors, a *pendant vertex* is a vertex with exactly one neighbor, and a *universal vertex* is a vertex adjacent to every other vertex of the graph. The neighborhood of the vertex v is denoted by $N_G(v)$. Two vertices v, w are *false twins* in G if they are nonadjacent and $N_G(v) = N_G(w)$, while they are *true twins* in G if they are false twins in $\overline{G}$. If H is a subgraph of G, we define $N_H(v) = N_G(v) \cap V(H)$.

Let $A, B \subseteq V(G)$. We say that A *is complete to* B if every vertex of A is adjacent to every vertex of B; and A *is anticomplete to* B if A is complete to B in $\overline{G}$. If S is any set, we denote the cardinality of S by $|S|$.

A class of graphs $\mathcal{G}$ is *hereditary* if every induced subgraph of every member of $\mathcal{G}$ belongs to $\mathcal{G}$. Given two graphs G and H, the graph G is H*-free* if G contains no induced H. If $\mathcal{H}$ is a collection of graphs, G is said to be $\mathcal{H}$*-free* if G is H-free for each $H \in \mathcal{H}$.

The set $X \subseteq V(G)$ is a *complete set* (resp. *stable set*) of G if the elements of X are pairwise adjacent (resp. nonadjacent). A *clique* of G is a complete set that is maximal under inclusion.

A *chord* of a cycle (resp. path) is an edge joining two nonconsecutive vertices of the cycle (resp. path). We denote the chordless path on n vertices by P_n, the chordless cycle on n vertices by C_n, and the complete graph on n vertices by K_n. K_1 is called *trivial* and K_3 is called the *triangle*. A *star* is the complete bipartite graph $K_{1,n}$ for some n. For any graph G, we denote by G^+ the graph that arises from G by adding a universal vertex, and by G^* the graph that arises from G by adding an isolated vertex.

Some small graphs to be referred in the sequel are depicted in Figure 1.

A graph G is a *circle graph* if it is the intersection graph of a family $L = \{C_v\}_{v \in V(G)}$ of chords of a circle (i.e., for each $v, w \in V(G)$, $vw \in E(G)$ if and only if $v \neq w$ and $C_v \cap C_w \neq \emptyset$). L is called a *circle model* of G. Circle graphs were introduced by Even and Itai in [12] to solve a problem of queues and stacks posed by Knuth in [21].

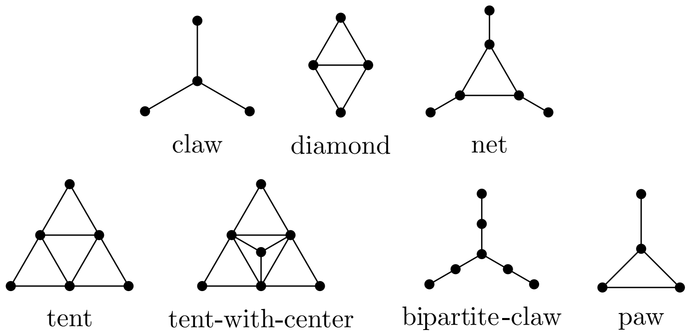

Figure 1: Some small graphs

Naji [25] characterized circle graphs in terms of the solvability of a system of linear equations, yielding a polynomial-time recognition algorithm for this class. Different polynomial-time recognition algorithms for circle graphs, strongly based on the notion of split decomposition, were presented in the literature. The best one has a quadratic time complexity and is due to Spinrad [28].

The *local complement* of a graph G with respect to a vertex $u \in V(G)$ is the graph $G * u$ that arises from G by replacing the induced subgraph $G[N_G(u)]$ by its complement. Two graphs G and H are *locally equivalent* if and only if G arises from H by a finite sequence of local complementations.

Theorem 1. [4] *The class of circle graphs is closed by local complementations.*

Moreover, Bouchet gave the following characterization of circle graphs in terms of forbidden induced subgraphs and local equivalence.

Theorem 2. [4] *Let G be a graph. Then, G is a circle graph if and only if no graph locally equivalent to G contains W_5, W_7, or BW_3 as induced subgraph (see Figure 2).*

In [8] a superclass of circle graphs (denoted as Bouchet graphs) is defined. A graph G is Bouchet if and only if no induced subgraph of G is locally equivalent to W_5, W_7, or BW_3. The list of 33 minimal forbidden induced subgraphs for this class is obtained using a computer, closing under local complementation the graphs W_5, W_7 and BW_3. Clearly, the graphs of this family are also minimal forbidden subgraphs for circle graphs. But this list

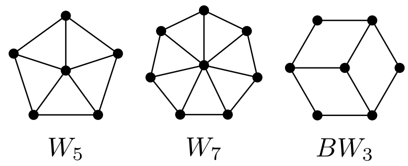

Figure 2: Graphs W_5, W_7 and BW_3

is not enough to characterize circle graphs completely. In the same work it is shown that circle graphs are a proper subclass of Bouchet graphs.

Recently, Geelen and Oum [14] gave a new characterization of circle graphs in terms of pivoting. The result of pivoting a graph G with respect to an edge uv is the graph $G \times uv = G * u * v * u$ (where $*$ stands for local complementation). A graph G' is *pivot-equivalent* to G if G' arises from G by a sequence of pivoting operations. They proved, with the aid of a computer, that G is a circle graph if and only if each graph that is pivot-equivalent to G contains none of 15 prescribed induced subgraphs.

In spite of the mentioned works, there are not known characterizations of circle graphs only by forbidden induced subgraphs, i.e. not involving additionally the notions of local equivalence or pivoting operations. In this paper, we present some results in this direction, providing forbidden induced subgraphs characterizations of circle graphs within different graph classes (a similar approach in order to find partial characterizations of circular-arc graphs by minimal forbidden induced subgraphs was developed by us in [2]). In Section 2 we present the main result of this paper, namely, we characterize circle graphs within linear domino graphs, in a constructive way. In Section 3, the same task is done within two superclasses of cographs (namely, P_4-tidy graphs and tree-cographs), by using the forbidden induced subgraphs characterization of permutation graphs. Finally, in the last section, we completely characterize by minimal forbidden induced subgraphs the class of *unit Helly circle graphs*, which are those circle graphs having a model whose chords have all the same length, are pairwise different, and satisfy the Helly property.

For definitions and notions not introduced in this section and used throughout the paper, the reader is referred to [5].

2. Linear domino graphs

A graph G is *domino* if all its vertices belong to at most two cliques. If, in addition, each of its edges belongs to at most one clique, then G is a *linear domino graph.* Linear domino graphs coincide with {claw,diamond}-free graphs [20]. Linear domino graphs have also a nice property related with clique coverings [22].

In this section we will characterize circle graphs by minimal forbidden induced subgraphs within the class of linear domino graphs, using a constructive way.

Let G_1 and G_2 be two graphs such that $|V(G_i)| \geq 3$, for each $i = 1, 2$, and assume that $V(G_1) \cap V(G_2) = \emptyset$. Let v_i be a distinguished vertex of G_i, for each $i = 1, 2$. The *split composition* of G_1 and G_2 with respect to v_1 and v_2 is the graph $G_1 * G_2$ whose vertex set is $V(G_1 * G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$ and whose edge set is $E(G_1 * G_2) = E(G_1 - \{v_1\}) \cup E(G_2 - \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}$. The vertices v_1 and v_2 are called the *marker vertices.* We say that G has a *split decomposition* if there exist two graphs G_1 and G_2 with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 * G_2$ with respect to some pair of marker vertices. If so, G_1 and G_2 are called the *factors* of the split decomposition. Notice that G_1 and G_2 are induced subgraphs of G. Those graphs that do not have a split decomposition are called *prime graphs.* Notice that if any of the factors of a split decomposition admits a split decomposition we can continue the process until every factor is prime, a star or a complete graph. The resulting decomposition into prime graphs, stars and complete graphs might not be unique. Nevertheless, in [7] it is proved that if the number of factors is minimum then the decomposition is unique (up to reordering of the factors). Bouchet proved that circle graphs are closed under split composition.

Theorem 3. [3] *Let G be a graph that has a split decomposition $G = G_1 * G_2$. Then, G is a circle graph if and only if both G_1 and G_2 are circle graphs.*

The operation of edge subdivision in a graph G consists on selecting an edge uv of G and replacing it with the path uzv, where z is a new vertex. As a consequence of Theorem 1, we can prove the following result.

Theorem 4. *Let G be a graph. If G is not a circle graph, then any graph H that arises from G by edge subdivisions is not a circle graph.*

Proof. Suppose that H arises from G by edge subdivisions. So, H is be obtained from G by replacing some edges of G by paths of length at least two. It is easy to see that if local complementation is applied successively on each internal vertex of these paths, traversing one path at a time (in any of the two possible directions each), the graph H' that arises from these operations contains an induced G. Since G is not a circle graph and the class is hereditary, H' is not a circle graph. Hence, by Theorem 1, H is not a circle graph.□

A *prism* is a graph that consists of two disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ linked by three vertex disjoint paths P_1, P_2, P_3 where P_i links a_i and b_i for $i = 1, 2, 3$, and such that all the internal vertices of P_1, P_2 and P_3 have degree 2. The graph $\overline{C_6}$ is a prism where P_1, P_2 and P_3 have just one edge each. This graph is locally equivalent to W_5, so by Theorem 2, $\overline{C_6}$ is not a circle graph. Besides, since every prism arises from $\overline{C_6}$ by edge subdivision, Theorem 4 implies that prisms are not circle graphs.

The following theorem characterizes those linear domino graphs that are circle graphs.

Theorem 5. *Let G be a linear domino graph. Then, G is a circle graph if and only if G contains no induced prisms.*

Proof. The "only if" part follows immediately from Theorem 4 and the fact that the class of circle graphs is hereditary. Suppose now that G is a linear domino graph not containing induced prisms. We shall prove that G is a circle graph. Consider the factors of a split decomposition of G into prime graphs, stars and complete graphs. It is easy to see that stars and complete graphs are circle graphs. Therefore, by Theorem 3, we may suppose that G is a prime graph. Since a graph is a circle graph if and only if each of its connected components is a circle graph, we can assume also that G is connected. Since trees are circle graphs, we can suppose that G contains at least one chordless cycle. Consider a chordless cycle of G of maximum length, say $C = v_1v_2 \dots v_nv_1$, and let $X \subseteq V(G)$ be the set of all the vertices having at least one neighbor in C. We will prove that actually $V(C) \cup X = V(G)$ and that G is a circle graph. We will split the proof into three cases: $n = 3$, $n = 4$ or 5, and $n \geq 6$. (From now on, all the operations between indexes should be understood modulo n.)

Case 1: $n = 3$. In this case we will prove that G is isomorphic to C. Suppose by the way of contradiction that G is not isomorphic to C and thus, since G is connected, $X \neq \emptyset$. If v is a vertex in X, it necessarily has either one or three neighbors on C, otherwise G would contain an induced diamond. Besides, if $v, w \in X$ with $|N_C(v)| = 1$ (say $N_C(v) = \{v_1\}$) and $|N_C(w)| = 3$, then they are not adjacent. Because, if they were adjacent, then v, w, v_1, v_2 would induce a diamond in G. On one hand, if $v, w \in X$ and $|N_C(v)| = |N_C(w)| = 1$, then they are adjacent if and only if $N_C(v) = N_C(w)$. Indeed, if $N_C(v) = N_C(w) = \{v_i\}$ and v and w were not adjacent, then the vertices v, w, v_i, v_{i+1} would induce a claw, a contradiction. Conversely, if $N_C(v) = \{v_i\}$, $N_C(w) = \{v_{i+1}\}$ and $vw \in E(G)$, the set of vertices $\{v, w, v_i, v_{i+1}\}$ would induce a C_4. This is a contradiction, because we are assuming that C is a chordless cycle of maximum length. On the other hand, if $v, w \in X$ and $|N_C(v)| = |N_C(w)| = 3$, then v and w are adjacent because otherwise v, w, v_1, v_2 would induce a diamond. As a consequence of these observations, it follows that $X = Q_1 \cup Q_2 \cup Q_3 \cup Q$ where Q_1, Q_2, Q_3, Q are complete sets, Q_i is complete to v_i and anticomplete to $V(C) \setminus \{v_i\}$ for every $i = 1, 2, 3$, Q is complete to $V(C)$, and Q_1, Q_2, Q_3, Q are pairwise anticomplete. We will prove that Q_1, Q_2, Q_3, Q (when they are non-empty) belong to different connected components of $G - V(C)$ because of the maximality of C. By the way of contradiction, let P be a path in $G - V(C)$ of minimum length joining two vertices of X that belong to different sets of the partition $X = Q_1 \cup Q_2 \cup Q_3 \cup Q$. By construction, P has length at least 2 and has no internal vertex in $V(C) \cup X$. By symmetry, we just have to consider two cases: the extremes of P are either $w_i \in Q_i$ and $w_j \in Q_j$ with $i \neq j$, or $w_i \in Q_i$ and $w \in Q$. In the former case, $V(P) \cup \{v_i, v_j\}$ would induce a chordless cycle of length at least five. In the latter case, $V(P) \cup \{v_i\}$ would induce a chordless cycle of length at least four. Both contradictions prove that indeed Q_1, Q_2, Q_3, Q (if non-empty) belong to different connected components of $G - V(C)$ that will be denote by R_1, R_2, R_3, R, respectively. Since G is a prime graph, $Q_i = \emptyset$ for all $i = 1, 2, 3$. Otherwise, $V(R_i) \cup \{v_i, v_{i+1}\}$ and $V(G) \setminus V(R_i)$ form a split decomposition of G, with v_{i+1} and v_i as marker vertices, respectively. For a similar reason, $Q = \emptyset$. Thus, $V(G) = \{v_1, v_2, v_3\}$ and G is clearly a circle graph.

Case 2: $n = 4$ or 5. Since G is a linear domino graph, $|N_C(v)| = 2$ for every vertex v belonging to X and the two neighbors are consecutive in C. We will prove that if $v, w \in X$, then $vw \in E(G)$ if and only if $N_C(v) = N_C(w)$. Suppose that $N_C(v) \neq N_C(w)$. On one hand, if $N_C(v) \cap N_C(w) = \{z\}$ and $vw \in E(G)$, then $G[\{v, w, y, z\}]$ would be isomorphic to a diamond for each $y \in (N_C(v) \cup N_C(w)) \setminus \{z\}$, contradiction. On the other hand, if $N_C(v) \cap N_C(w) = \emptyset$ and $vw \in E(G)$, then $C \cup \{v, w\}$ would induce a prism in G, another contradiction. So, if $N_C(v) \neq N_C(w)$, then v and w are nonadjacent. Finally, if $N_C(v) = N_C(w) = \{y, z\}$, then v and w are adjacent, otherwise $\{v, w, y, z\}$ would induce a diamond, a contradiction. Hence $X = Q_1 \cup \cdots \cup Q_n$, where each Q_i is a complete set and $N_C(x) = \{v_i, v_{i+1}\}$ for every $x \in Q_i$. We will prove that the non-empty Q_i's belong to a different connected component of $G - V(C)$. By the way of contradiction, consider path P in $G - V(C)$ of minimum length joining two vertices $w_i \in Q_i$ and $w_j \in Q_j$ with $i \neq j$. By symmetry, we just have to consider two cases: $j = i+1$ and $j = i+2$. By construction, P has at least two edges and has no internal vertex in $V(C) \cup X$. In the first case, $V(P) \cup (V(C) \setminus \{v_{i+1}\})$ induces a cycle of length strictly greater than n. In the second case, $V(P) \cup V(C)$ induces a prism whose triangles are $\{w_i, v_i, v_{i+1}\}$ and $\{w_{i+2}, v_{i+2}, v_{i+3}\}$. Both contradictions prove that indeed each non-empty Q_i belongs to a different connected component R_i of $G - V(C)$. Since G is prime, it follows that if Q_i is non-empty then $|V(R_i)| = 1$. Otherwise, let $w_i \in Q_i$. Then, $V(R_i) \cup \{v_i\}$ and $(V(G) \setminus V(R_i)) \cup \{w_i\}$ would be a split decomposition of G, with v_i and w_i as marker vertices, respectively.

So, G consists of C and a (possibly empty) stable set X with at most one vertex w_i for each $1 = 1, \ldots, n$, whose only neighbors in G are v_i and v_{i+1}. It is easy to build a circle model for G.

Case 3: $n \geq 6$. First, notice that, since G is a linear domino graph, every vertex $v \in X$ satisfies either $N_C(v) = \{v_i, v_{i+1}\}$ or $N_C(v) = \{v_i, v_{i+1}, v_{i+k}, v_{i+k+1}\}$ with $3 \leq k \leq n-3$. We will call the first kind of vertices 2-vertices and the second kind of vertices 4-vertices. It can be easily proved, as above, that if v and w are 2-vertices, then v and w are adjacent if and only if $N_C(v) = N_C(w)$. Let us see that if $v \in X$ is a 2-vertex and $w \in X$ is a 4-vertex, then v is adjacent to w if and only if $N_C(v) \subseteq N_C(w)$. Let $N_C(w) = \{v_i, v_{i+1}, v_{i+k}, v_{i+k+1}\}$. Suppose first that $vw \in E(G)$. Since w is not the center of a claw, v should be adjacent to at least one vertex of each pair of nonadjacent neighbors of w. Besides, since $N_C(v)$ consists of two consecutive vertices of C, they should be either $\{v_i, v_{i+1}\}$ or $\{v_{i+k}, v_{i+k+1}\}$. Conversely, suppose that $N_C(v) \subseteq N_C(w)$. Again, since $N_C(v)$ consists of two consecutive vertices of C, then $N_C(v)$ should be either $\{v_i, v_{i+1}\}$ or $\{v_{i+k}, v_{i+k+1}\}$. Since G is diamond-free, v and w must be adjacent.

Let v and w be two 4-vertices. We assert that $|N_C(v) \cap N_C(w)| \in \{0, 1, 2\}$ and that $vw \in E(G)$ if and only if $N_C(v) \cap N_C(w)$ consists of two consecutive vertices of C. If $N_C(v) \cap N_C(w)$ contains two nonadjacent vertices x and y, then v and w should be nonadjacent, otherwise $\{x, y, v, w\}$ would induce a diamond in G. On the other hand, if $N_C(v) \cap N_C(w)$ contains two adjacent vertices x and y, then v and w should be adjacent, otherwise $\{x, y, v, w\}$

would induce a diamond in G. Therefore, v and w can share neither three nor four neighbors, and the "if" of the second part of our assertion holds. Conversely, suppose $vw \in E(G)$. Since w is not the center of a claw, v should be adjacent to at least one vertex of any pair of nonadjacent neighbors of w, so $N_C(v) \cap N_C(w)$ contains two adjacent vertices. If $N_C(v) \cap N_C(w)$ contained two nonadjacent vertices x and y, then $\{x, y, v, w\}$ would induce a diamond in G, so $N_C(v) \cap N_C(w)$ consists exactly of two consecutive vertices of C.

Therefore, X is a disjoint union of the sets of vertices $Q_1, \cdots, Q_n, Q$, where the vertices in Q are the 4-vertices and the vertices in $Q_1 \cup \cdots \cup Q_n$ are the 2-vertices such that $N_C(x) = \{v_i, v_{i+1}\}$ for each $x \in Q_i$. Each Q_i is a complete set and anticomplete to Q_j if $i \neq j$. Since two 4-vertices share at most two neighbors in C, in particular there are no two vertices in Q with the same neighbors in C. Therefore, the set Q is a subset of $\{q_{i,j} : 1 \leq i < j \leq n, i+3 \leq j \leq n+i-3\}$, where $N_C(q_{i,j}) = \{v_i, v_{i+1}, v_j, v_{j+1}\}$, $q_{i,j}$ is complete to Q_i and Q_j and anticomplete to Q_k for $k \neq i, j$, and $q_{i,j}q_{i',j'} \in E(G)$ if and only if $|\{i, j\} \cap \{i', j'\}| = 1$. Notice that no vertex $q_{i,j}$ of Q has a neighbor z not in $C \cup X$, otherwise $\{q_{i,j}, v_i, v_j, z\}$ would induce a claw in G, a contradiction.

We will prove now that the non-empty Q_i's belong to different connected components of $G - (V(C) \cup Q)$. By the way of contradiction, let P be a path in $G - (V(C) \cup Q)$ of minimum length joining two vertices $w_i \in Q_i$ and $w_j \in Q_j$ with $i \neq j$. By construction, P has length at least two and has no internal vertices that belong to $V(C) \cup X$. On one hand, if $|N_C(w_i) \cap N_C(w_j)| = 1$, then G would contain a chordless cycle of length greater than n, a contradiction. On the other hand, if $N_C(w_i) \cap N_C(w_j) = \emptyset$, then G would contain an induced prism, also a contradiction. So, indeed each of the non-empty Q_i's belong to a different connected component R_i of $G - (V(C) \cup Q)$. Since G is prime, it follows that if Q_i were non-empty then $|V(R_i)| = 1$. Otherwise, let $w_i \in Q_i$. Then $V(R_i) \cup \{v_i\}$ and $(V(G) \setminus V(R_i)) \cup \{w_i\}$ would be a split decomposition of G, with v_i and w_i as marker vertices, respectively.

Consider now two nonadjacent 4-vertices v and w. Then, the edges of C with either both endpoints in $N_C(v)$ (say v-edges) or both endpoints in $N_C(w)$ (say w-edges) are exactly four. We will prove that traversing the edges of C in clockwise order, v-edges and w-edges do not alternate, otherwise G would contain an induced prism. Suppose by the way of contradiction that the edges in clockwise order are e_1, e_2, e_3, e_4 where e_1, e_3 are v-edges and e_2, e_4 are w-edges. Either e_1 and e_2, or e_2 and e_3 are nonconsecutive in C, since e_1 and e_3 are at least two edges apart in C. Suppose without loss of generality that e_1 and e_2 are nonconsecutive in C. Let z_1^i and z_2^i be the endpoints of e_i in clockwise order. Then, by removing vertices z_2^3 and z_1^4 and the clockwise path in C linking them from $G[V(C) \cup \{v, w\}]$, a prism arises: the triangles are $\{z_1^1, z_2^1, v\}$ and $\{w, z_1^2, z_2^2\}$; w is linked with z_1^1 via z_2^4 and the path in C joining z_2^4 and z_1^1 (they might be the same vertex); z_2^1 and z_1^2 are different and linked by a path in C; z_2^2 an v are linked via z_1^3 and the path in C joining z_2^2 and z_1^3 (they might be the same vertex).

Next, we will build a circle model for G. Draw a circle $\mathcal{C}$ and mark on $\mathcal{C}$, in clockwise order, the following points: c_n, a_1, $f_{n,3}$, $\ldots$, $f_{n,n-3}$, b_n, d_n, c_1, a_2, $f_{1,4}$, $\ldots$, $f_{1,n-2}$, b_1, d_1, c_2, a_3, $f_{2,5}$, $\ldots$, $f_{2,n-1}$, b_2, d_2, $\ldots$, c_{n-1}, a_n, $f_{n-1,2}$, $\ldots$, $f_{n-1,n-4}$, b_{n-1}, d_{n-1}. Finally, draw the chords a_ib_i for $i = 1, \ldots, n$, the chord c_id_i for each i in $\{1, \ldots, n\}$ such that Q_i is non-empty, and the chord $f_{i,j}f_{j,i}$ for each i, j in $\{1, \ldots, n\}$ such that $q_{i,j} \in Q$.$\square$

A *theta* is a graph arising from $K_{2,3}$ by edge subdivision. Chudnovsky and Kapadia [6] gave a polynomial-time algorithm that decides whether a graph contains a theta or a prism as induced subgraphs. Since linear domino graphs contain no induced theta, the characterization above and the existence of polynomial-time algorithms for recognizing circle graphs imply alternative polynomial-time algorithms to decide the existence of an induced theta or prism restricted to linear domino graphs. Interestingly enough, the problem of deciding whether a graph contains an induced prism is NP-complete in general [23].

3. Superclasses of cographs

Cographs are the P_4-free graphs. It is well-known that cographs are circle graphs. Moreover, every nontrivial cograph is either disconnected or the join of two smaller cographs. (This fact was discovered independently by several authors since the 1970s; early references include [27].) We are interested in the characterization of circle graphs within two superclasses of cographs: P_4-tidy graphs and tree-cographs. To this end, we will use a forbidden induced subgraph characterization of the class of permutation graphs.

A graph is said to be a *comparability* graph if its edges can be transitively oriented. In [13], a characterization of comparability graphs by means of a list of forbidden induced subgraphs is given. A graph G is a *permutation* graph if and only if G and $\overline{G}$ are comparability graphs [26]. Therefore, the characterization of comparability graphs in [13] leads immediately to a forbidden induced subgraph characterization of permutation graphs.

Theorem 6. [13] *A graph is a comparability graph if and only if it does not contain as an induced subgraph any graph in Figure 3 and its complement does not contain as an induced subgraph any graph in Figure 4.*

Corollary 7. *A graph G is a permutation graph if and only if G and $\overline{G}$ do not contain as an induced subgraph any graph in Figures 3 and Figure 4.*

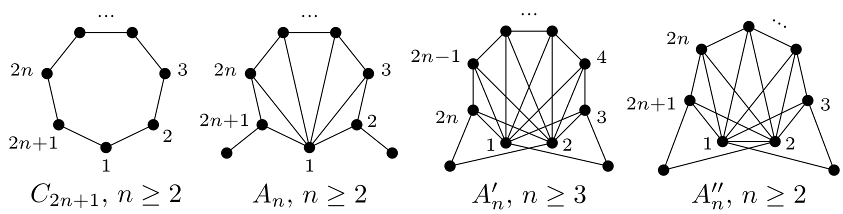

Figure 3: Some minimal forbidden induced subgraphs for comparability graphs.

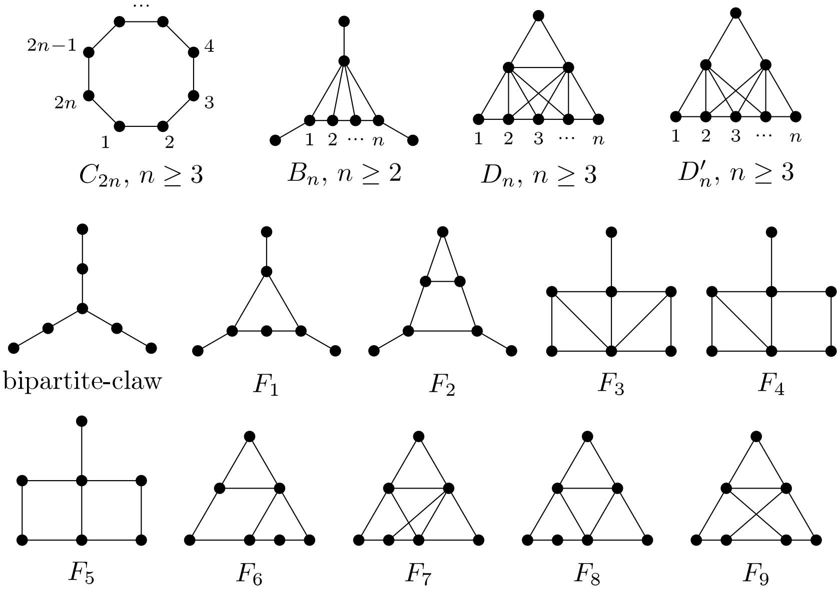

Figure 4: Some graphs whose complements are minimal forbidden induced subgraphs for comparability graphs.

Let G_1 and G_2 be two graphs and assume that $V(G_1) \cap V(G_2) = \emptyset$. The *disjoint union* of G_1 and G_2 is the graph $G_1 \cup G_2$ such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. We denote by $G_1 + G_2$

the *join graph* of G_1 and G_2, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Permutation graphs are exactly those circle graphs that have a circle model admitting an *equator*, i.e. an additional chord meeting all the chords of the model [16, p. 252]. Equivalently, G^+ is a circle graph if and only if G is a permutation graph. The following result is an immediate consequence.

Lemma 8. *The join $G = G_1 + G_2$ is a circle graph if and only if both G_1 and G_2 are permutation graphs.*

3.1. P_4-tidy graphs

Let G be a graph and let A be a vertex set that induces a P_4 in G. A vertex v of G is said a *partner* of A if $G[A \cup \{v\}]$ contains at least two induced P_4's. Finally, G is called *P_4-tidy* if each vertex set A that induces a P_4 in G has at most one partner [15].

The class of P_4-tidy graphs is an extension of the class of cographs and it contains many other graph classes defined by bounding the number of P_4's according to different criteria; e.g., P_4-sparse graphs [17], P_4-lite graphs [18], and P_4-extendible graphs [19].

A *spider* [17] is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. The triple (S, C, R) is called the *spider partition*. Clearly, the complement of a thin spider is a thick spider, and vice versa. A *fat spider* is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$. The following theorem characterizes the structure of P_4-tidy graphs.

Theorem 9. [15] *Let G be a P_4-tidy graph with at least two vertices. Then, exactly one of the following conditions holds:*

1. *G is disconnected.*
2. *$\overline{G}$ is disconnected.*
3. *G is isomorphic to P_5, $\overline{P_5}$, C_5, a spider, or a fat spider.*

Before giving the next characterization, we state the following lemma whose proof is straightforward.

Lemma 10. *Let G be a graph and let H be a graph obtained from G by adding either a pendant vertex, or a true or false twin of a vertex. Then, H is a circle graph if and only if G is a circle graph.*

Bandelt and Mulder have shown in [1] that a graph is distance hereditary if and only if it can be generated by the operations pendant vertex, true twin and false twin from a single vertex. Consequently, Lemma 10 implies that every distance-hereditary graph is a circle graph. This fact is already mentioned in [5].

Theorem 11. *Let G be a P_4-tidy graph. Then, G is a circle graph if and only if G contains no W_5, net$^+$, tent$^+$, or tent-with-center as induced subgraph.*

Proof. It is easy to see that net$^+$, tent$^+$, and tent-with-center are not circle graphs. Since the class of circle graphs is hereditary, a circle graph contains no induced net$^+$, tent$^+$, or tent-with-center.

Conversely, let G be a P_4-tidy graph that is not a circle graph. Then, G contains some induced graph H that is minimally not circle; i.e., H is not a circle graph but all proper induced subgraphs of H are circle graphs. Because of the minimality, H is connected. Suppose first that $\overline{H}$ is disconnected; i.e., $H = H_1 + H_2$ for some graphs H_1 and H_2. By Lemma 8, since H is not a circle graph, H_1 or H_2 is not a permutation graph. By Corollary 7, H_1 or H_2 contains an induced C_5, net, or tent. Thus, $H = H_1 + H_2$ contains an induced W_5, net$^+$, or tent$^+$. By minimality, $H = W_5$, net$^+$, or tent$^+$. Suppose, on the contrary, that $\overline{H}$ is connected. By Theorem 9, since H is a P_4-tidy graph, either H is C_5, P_5, $\overline{P_5}$, a spider, or a fat spider. Since H is not a circle graph, H is different from C_5, P_5, and $\overline{P_5}$. Thus, H is a spider or a fat spider. By Lemma 10 and the minimality, H has no true or false twins, so H is not a fat spider. We conclude that H is a spider. Let (S, C, R) be the spider partition of H. By Lemma 10 and the minimality, H is necessarily a thick spider with $|S| \geq 3$. Since tent is a circle graph, either $|S| \geq 4$ or $R \neq \emptyset$. In both cases, H contains an induced tent-with-center and, by minimality, $H =$ tent-with-center.□

3.2. Tree cographs

Tree-cographs [29] are another generalization of cographs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph. It is immediate from the definition that, if G is a tree-cograph, then G or $\overline{G}$ is disconnected, or G or $\overline{G}$ is a tree.

Theorem 12. *Let G be a tree-cograph. Then, G is a circle graph if and only if G contains no induced (bipartite-claw)$^+$ and no induced co-(bipartite-claw).*

Proof. It is easy to see that bipartite-claw$^+$ and co-(bipartite-claw) are not circle graphs and thus a circle graph contains none of those graphs as induced subgraph. Conversely, let G be a tree-cograph that is not a circle graph. Therefore, there exists some connected component H of G that is not a circle graph. Notice that H cannot be a tree because trees are circle graphs. Since H is a tree-cograph and H is connected, $\overline{H}$ is disconnected or $\overline{H}$ is a tree. Suppose first that $\overline{H}$ is disconnected. Then, $H = H_1 + H_2$ for some graphs H_1 and H_2. By Lemma 8, we can assume without loss of generality that H_1 is not a permutation graph. Corollary 7 implies that H_1 would contain an induced bipartite-claw, and so $H = H_1 + H_2$ would contain an induced (bipartite-claw)$^+$. Finally, consider the case when $\overline{H}$ is a tree. Since H is not a circle graph, in particular it is not a permutation graph. By Corollary 7, H contains an induced co-(bipartite-claw).□

4. Unit Helly circle graphs

A graph G is a *unit circle* graph if it admits a circle model in which all the chords have the same length. This class coincides with the class of unit circular-arc graphs (i.e., the intersection graphs of a family of arcs on a circle, all of the same length) [10]. Tucker gave a characterization by minimal forbidden induced subgraphs for this class [30]. Recently, linear and quadratic-time recognition algorithms for this class have been proposed [24, 11].

The concept of *Helly circle graph* is due to Durán [10]. A graph belongs to this class if it has a circle model whose chords are pairwise different and satisfy the Helly property (i.e., every subset of pairwise intersecting chords has a common point). In [10], it was conjectured that a circle graph is a Helly circle graph if and only if it is diamond-free. This conjecture was recently settled affirmatively in [9], yielding a polynomial-time recognition algorithms for Helly circle graphs.

In the theorem below we completely characterize unit Helly circle graphs.

Theorem 13. *Let G be a graph. Then, the following assertions are equivalent:*

1. *G is a unit Helly circle graph.*
2. *G contains no induced claw, paw, diamond, or C_n^* for any $n \geq 3$.*
3. *G is a chordless cycle, a complete graph, or a disjoint union of chordless paths.*

Proof. Let us consider the case when G is triangle-free. Suppose first that 1 holds. Since G is a unit circle graph, G is a unit circular-arc graph. Thus, G contains no induced claw or C_n^* for any $n \geq 4$ [30]. This proves $1 \Rightarrow 2$ (in the case when G is triangle-free). Suppose now that 2 holds. If G has no cycles, then each connected component of G is a claw-free tree, i.e., G is the disjoint union of chordless paths. So, assume that G has some cycle. Since G is triangle-free, the shortest cycle H of G is a chordless cycle of length at least 4. Since G contains no induced claw, triangle, or C_n^* for any $n \geq 4$, $G = H$. We conclude that $2 \Rightarrow 3$. Finally, it is easy to build unit Helly circle models of chordless cycles and of disjoint unions of chordless paths. Consequently, $3 \Rightarrow 1$ also holds.

Let us now consider the case when G is not triangle-free. Suppose that 1 holds and let $\mathcal{L} = \{L_i\}_{i=1}^n$ be a unit Helly model of G on a circle $\mathcal{C}$, where $n = |V(G)|$. If two different chords L_1 and L_2 on $\mathcal{C}$ have the same length, then L_1 and L_2 are diameters of $\mathcal{C}$ or both of them are tangent to a circle $\mathcal{C}'$ concentric with $\mathcal{C}$. Since G is not triangle-free, we can assume that L_1, L_2, and L_3 are three pairwise intersecting chords and, since $\mathcal{L}$ has the Helly property, there is a point $P \in L_1 \cap L_2 \cap L_3$. We claim that L_1, L_2, and L_3 are diameters of $\mathcal{C}$. Otherwise, L_1, L_2, and L_3 would be three different tangents to a circle $\mathcal{C}'$ through P and this would lead to a contradiction, because it is well-known that there are at most two different tangents to a circle passing through a given point. Since all chords of $\mathcal{L}$ have all the same length, then $\mathcal{L}$ is a family of diameters of $\mathcal{C}$ and, therefore, G is a complete graph. We conclude that $1 \Leftrightarrow 3$ because complete graphs are clearly unit Helly circle graphs. Finally, given that G contains a triangle, it is straightforward that G is a complete graph if and only if G contains no induced C_3^*, paw, or diamond. (Notice that C_3^*, paw, and diamond are all the four-vertex graphs that contain the triangle as induced subgraph and that are not complete graphs.) We conclude that 2 $\Leftrightarrow$ 3 also holds.□

5. Further research

In [9] it is proved that Helly circle graphs are the circle graphs with no induced diamond. Consequently, Theorem 5 implies that, given a claw-free graph G, G is a Helly circle graph if and only if G does not contain any induced prisms. We think the tools used throughout the proof of the theorem might be either adapted or recycled in order to get Helly circle graphs thoroughly characterized by means of a list of forbidden induced subgraphs, without the assumption that the graph is a circle graph.

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