Exploring the complexity boundary between coloring and list-coloring

Flavia Bonomo [1]

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina

Guillermo Durán [2]

Departamento de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Chile

Javier Marenco [3]

Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina

Abstract

Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs. However, the list-coloring problem is NP-complete for many subclasses of perfect graphs. In this work we explore the complexity boundary between vertex coloring and list-coloring on such subclasses of perfect graphs, where the former admits polynomial-time algorithms but the latter is NP-complete. Our goal is to analyze the computational complexity of coloring problems lying "between" (from a computational complexity viewpoint) these two problems: precoloring extension, μ-coloring, and (γ, μ)-coloring.

Key words: coloring, computational complexity, list-coloring

1 Introduction

A *coloring* of a graph $G = (V, E)$ is a function $f : V \to \mathbf{N}$ such that $f(v) \neq f(w)$ whenever $vw \in E$. A k-*coloring* is a coloring f such that $f(v) \leq k$ for every $v \in V$. The *vertex coloring problem* takes as input a graph G and a natural number k, and consists in deciding whether G is k-colorable or not. This well-known problem is a basic model for frequency assignment and resource allocation problems.

In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. One of such generalized models is the *list-coloring problem*, which considers a prespecified set of available colors for each vertex. Given a graph G and a finite list $L(v) \subseteq \mathbf{N}$ for each vertex $v \in V$, the list-coloring problem asks for a *list-coloring* of G, i.e., a coloring f such that $f(v) \in L(v)$ for every $v \in V$.

Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs [4]. However, the list-coloring problem is NP-complete for general perfect graphs, and is also NP-complete for many subclasses of perfect graphs, including split graphs [9], interval graphs [1,12], and bipartite graphs [9]. In this work we explore the complexity boundary between vertex coloring and list-coloring on such subclasses of perfect graphs, where the former admits polynomial-time algorithms but the latter is NP-complete. Our goal is to analyze the computational complexity of coloring problems lying "between" (from a computational complexity viewpoint) these two problems.

We consider the following particular cases of the list-coloring problem. The *precoloring extension* (PrExt) problem takes as input a graph $G = (V, E)$, a subset $W \subseteq V$, a coloring f' of W, and a natural number k, and consists in deciding whether G admits a k-coloring f such that $f(v) = f'(v)$ for every $v \in W$ or not [1]. In other words, a prespecified vertex subset is colored

Email addresses: `fbonomo@dm.uba.ar` (Flavia Bonomo), `gduran@dii.uchile.cl` (Guillermo Durán), `jmarenco@dc.uba.ar` (Javier Marenco).

[1] Partially supported by UBACyT Grant X184, PICT ANPCyT Grant 11-09112 and PID Conicet Grant 644/98 (Argentina), and CNPq under PROSUL project Proc. 490333/2004-4 (Brazil).

[2] Partially supported by FONDECyT Grant 1050747 and Millennium Science Nucleus "Complex Engineering Systems" (Chile), and CNPq under PROSUL project Proc. 490333/2004-4 (Brazil).

[3] Partially supported by UBACyT Grant X036, PICT ANPCyT Grant 11-09112 and PID Conicet Grant 644/98 (Argentina), and CNPq under PROSUL project Proc. 490333/2004-4 (Brazil).

beforehand, and our task is to extend this partial coloring to a valid k-coloring of the whole graph.

Given a graph G and a function $\mu : V \to \mathbf{N}$, G is *μ-colorable* if there exists a coloring f of G such that $f(v) \leq \mu(v)$ for every $v \in V$ [2]. This model arises in the context of classroom allocation to courses, where each course must be assigned a classroom which is large enough so it fits the students taking the course. We define here a new variation of this problem. Given a graph G and functions $\gamma, \mu : V \to \mathbf{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$, we say that G is *(γ, μ)-colorable* if there exists a coloring f of G such that $\gamma(v) \leq f(v) \leq \mu(v)$ for every $v \in V$.

The classical vertex coloring problem is clearly a special case of μ-coloring and precoloring extension, which in turn are special cases of (γ, μ)-coloring. Furthermore, (γ, μ)-coloring is a particular case of list-coloring. These observations imply that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete. We are, therefore, interested in the computational complexity of these problems over graph classes where vertex coloring is polynomially solvable and list-coloring is NP-complete.

2 Known results

A graph is an *interval graph* if it is the intersection graph of a set of intervals over the real line. A *unit interval graph* is the intersection graph of a set of intervals of length one. Since interval graphs are perfect, vertex coloring over interval and unit interval graphs is polynomially solvable. On the other hand, precoloring extension over unit interval graphs is NP-complete [12], implying that (γ, μ)-coloring and list-coloring are NP-complete over this class and over interval graphs.

A *split graph* is a graph whose vertex set can be partitioned into a complete graph K and an independent set I. A split graph is said to be *complete* if its edge set includes all possible edges between K and I. It is trivial to color a split graph in polynomial time, and it is a known result that precoloring extension is also solvable in polynomial time on split graphs [7], whereas list-coloring is known to be NP-complete even over complete split graphs [9].

A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets V_1 and V_2. A bipartite graph is said to be *complete* if its edge set includes all possible edges between V_1 and V_2. Again, the vertex coloring problem over bipartite graphs is trivial, whereas precoloring extension [6] and μ-coloring [2] are known to be NP-complete over bipartite graphs, implying that (γ, μ)-coloring and list-coloring over this class are also NP-complete. Moreover, list-coloring is NP-complete even over complete bipartite graphs [9].

For complements of bipartite graphs, precoloring extension can be solved in polynomial time [7], but list-coloring is NP-complete [8]. The same happens for *cographs*, graphs with no induced P_4 [9,7].

The *line graph* of a graph is the intersection graph of its edges. The edge coloring problem (equivalent to coloring the line graph) is NP-complete in general [5], but it can be solved in polynomial-time for complete graphs and bipartite graphs [10]. It is known that precoloring extension is NP-complete on line graphs of complete bipartite graphs $K_{n,n}$ [3], and list-coloring is NP-complete on line graphs of complete graphs [11].

A good survey on variations of the coloring problem can be found in [13].

3 New results

In this section we introduce new results related to the computational complexity of the previously mentioned coloring problems over the graph classes described in Section 2 and related classes.

Theorem 1 *The μ-coloring problem over interval graphs is NP-complete.*

This result implies that (γ, μ)-coloring over interval graphs also is NP-complete.

Theorem 2 *The (γ, μ)-coloring problem can be solved in polynomial time in complete bipartite graphs and complete split graphs.*

This result implies that μ-coloring and precoloring extension over complete bipartite graphs and complete split graphs can be solved in polynomial time. The algorithm for complete bipartite graphs relies on combinatorial arguments, whereas for complete split graphs integer programming techniques are employed.

Theorem 3 *The μ-coloring problem over split graphs is NP-complete.*

At this moment, this is the only class that we know where the computational complexity of μ-coloring and precoloring extension is different, unless P = NP.

Considering these coloring variations applied to edge coloring, we have the following results.

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
COMPLETE BIPARTITE	P	P	**P**	**P**	NP-c
BIPARTITE	P	NP-c	NP-c	NP-c	NP-c
COGRAPHS	P	P	P	?	NP-c
INTERVAL	P	NP-c	**NP-c**	NP-c	NP-c
UNIT INTERVAL	P	NP-c	?	NP-c	NP-c
SPLIT	P	P	**NP-c**	**NP-c**	NP-c
COMPLETE SPLIT	P	P	**P**	**P**	NP-c
LINE OF $K_{n,n}$	P	NP-c	**NP-c**	NP-c	NP-c
LINE OF K_n	P	**NP-c**	**NP-c**	**NP-c**	NP-c
COMPLEMENT OF BIPARTITE	P	P	?	?	NP-c

Table 1: Complexity table for coloring problems.

Theorem 4 *The μ-coloring problem over line graphs of complete graphs and complete bipartite graphs is NP-complete.*

Theorem 5 *The precoloring extension problem over line graphs of complete graphs is NP-complete.*

We summarize these results in Table 1. As this table shows, unless P = NP, μ-coloring and precoloring extension are strictly more difficult than vertex coloring (due to interval and bipartite graphs). On the other hand, list-coloring is strictly more difficult than (γ, μ)-coloring, due to complete split and complete bipartite graphs, and (γ, μ)-coloring is strictly more difficult than precoloring extension, due to split graphs. It remains as an open problem to know if there exists any class of graphs such that (γ, μ)-coloring is NP-complete and μ-coloring can be solved in polynomial time. Among the classes considered in this work, the candidate classes are COGRAPHS, UNIT INTERVAL and COMPLEMENT OF BIPARTITE.

Finally, we present some general results.

Theorem 6 *Let $\mathcal{F}$ be a family of graphs with minimum degree at least two. Then list-coloring, (γ, μ)-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Theorem 7 *Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, no connected component of G is complete, and for every vertex v of G, no connected component of $G \setminus v$ is complete. Then list-coloring, (γ, μ)-coloring, μ-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Please note that, since odd holes and antiholes satisfy the conditions of the theorems above, these theorems are applicable for many subclasses of perfect graphs.

References

[1] M. Biro, M. Hujter, and Zs. Tuza, Precoloring extension. I. Interval graphs, *Discrete Mathematics* **100**(1–3) (1992), 267–279.

[2] F. Bonomo and M. Cecowski, Between coloring and list-coloring: μ-coloring, *Electronic Notes in Discrete Mathematics* **19** (2005), 117–123.

[3] C.J. Colbourn, The complexity of completing partial Latin squares, *Annals of Discrete Mathematics* **8** (1984), 25–30.

[4] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169–197.

[5] I. Holyer, The NP-completeness of edge-coloring, *SIAM Journal on Computing* **10** (1981), 718–720.

[6] M. Hujter and Zs. Tuza, Precoloring extension. II. Graph classes related to bipartite graphs, *Acta Mathematica Universitatis Comenianae* **62**(1) (1993), 1–11.

[7] M. Hujter and Zs. Tuza, Precoloring extension. III. Classes of perfect graphs, *Combinatorics, Probability and Computing* **5** (1996), 35–56.

[8] K. Jansen, The optimum cost chromatic partition problem, *Lecture Notes in Computer Science* **1203** (1997), 25–36.

[9] K. Jansen and P. Scheffler, Generalized coloring for tree-like graphs, *Discrete Applied Mathematics* **75** (1997), 135–155.

[10] D. König, Über graphen und ihre anwendung auf determinantentheorie und mengenlehre, *Mathematische Annalen* **77** (1916), 453–465.

[11] M. Kubale, Some results concerning the complexity of restricted colorings of graphs, *Discrete Applied Mathematics* **36** (1992), 35–46.

[12] D. Marx, *Precoloring extension on unit interval graphs*, manuscript, 2004.

[13] Zs. Tuza, Graph colorings with local constraints – a survey, *Discussiones Mathematicae. Graph Theory* **17** (1997), 161–228.