On minimal forbidden subgraph characterizations of balanced graphs [1]

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Abstract

A $\{0,1\}$-matrix is balanced if it contains no square submatrix of odd order with exactly two 1's per row and per column. Balanced matrices lead to ideal formulations for both set packing and set covering problems. Balanced graphs are those graphs whose clique-vertex incidence matrix is balanced.

While a forbidden induced subgraph characterization of balanced graphs is known, there is no such characterization by *minimal* forbidden induced subgraphs. In this work we provide minimal forbidden induced subgraph characterizations of balanced graphs restricted to some graph classes which also lead to polynomial time or even linear time recognition algorithms within the corresponding subclasses.

Keywords: balanced graphs, line graphs, P_4-tidy graphs, paw-free graphs, perfect graphs

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1 Introduction

Balanced matrices are those $\{0,1\}$-matrices not having a square submatrix of odd order with exactly two 1's per row and per column. Balanced matrices have remarkable properties studied in polyhedral combinatorics. Most notably, if A is balanced, then the fractional set packing polytope $P(A) = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{1}, \mathbf{0} \leq x \leq \mathbf{1}\}$ and the fractional set covering polytope $Q(A) = \{x \in \mathbb{R}^n \mid Ax \geq \mathbf{1}, \mathbf{0} \leq x \leq \mathbf{1}\}$ are both integral (i.e., all their extreme points have integer coordinates) [8].

A $\{0,1\}$-matrix A is called *perfect* if and only if $P(A)$ is integral, and a graph is *perfect* if and only if its clique-matrix is perfect [6]. A *clique* Q in a graph $G = (V, E)$ is an inclusion-wise maximal subset of pairwise adjacent vertices. Given an enumeration $Q_1, \dots, Q_k$ of all cliques of G and an enumeration $v_1, \dots, v_n$ of all vertices of G, a *clique-matrix* of G is the $k \times n$ $\{0,1\}$-matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $v_j \in Q_i$. The clique-matrix of a graph is unique up to permutations of rows and/or columns.

Some years ago, the minimal forbidden induced subgraphs of perfect graphs were characterized [5], settling affirmatively a conjecture posed more than 40 years before by Berge [2]. The minimal forbidden induced subgraphs of perfect graphs are the chordless cycles of odd length having at least 5 vertices, called *odd holes* C_{2k+1}, and their complements, the *odd antiholes* $\overline{C}_{2k+1}$.

In analogy to perfect graphs, *balanced graphs* were defined to be those graphs whose clique-matrix is balanced [7]. Since balanced matrices are also perfect, the balanced graphs form a subclass of the class of perfect graphs. Balanced graphs were characterized by means of forbidden induced subgraphs in [3]. For a graph $G = (V, E)$ and $W \subseteq V$, let $N(W) = \bigcap_{w \in W} N(w)$ and use $N(e)$ as shorthand for $N(\{u, v\})$ for an edge $e = uv$. An *unbalanced cycle* of G is an odd cycle $C = (V', E')$ such that, for each edge $e \in E'$, there exists a (possibly empty) complete subgraph W_e of G such that $W_e \subseteq N(e) \setminus V'$ and $N(W_e) \cap N(e) \cap V' = \emptyset$. Note that the subsets W_e and W_f for different edges $e, f \in E'$ may overlap. An *extended odd sun* is a graph G with an unbalanced cycle C such that $V = V' \cup \bigcup_{e \in E'} W_e$ and $|W_e| \leq |N(e) \cap V'|$ for each edge e of C. The smallest extended odd suns are C_5 and the pyramids shown in Figure 2. Extended odd suns generalize odd suns, and can have a rather involved structure (cf. [3]).

The characterization of balancedness by forbidden subgraphs is as follows.

Theorem 1.1 ([3]) *A graph is balanced if and only if it has no unbalanced cycle, or, equivalently, if and only if it contains no induced extended odd sun.*

However, the above characterization is not by *minimal* forbidden induced subgraphs because some extended odd suns contain some other extended odd suns as proper induced subgraphs, as Figure 1 shows.

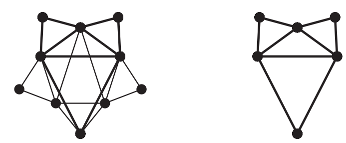

Fig. 1. On the left, an extended odd sun that is not minimal. Bold lines correspond to the edges of a proper induced extended odd sun, depicted on the right.

Thus, the above characterization forbids some subgraphs which are not essential to forbid. We address the problem to find the *minimal forbidden induced subgraphs*, i.e., those graphs that are not balanced but all their proper induced subgraphs are balanced. We present minimal forbidden induced subgraph characterizations of balanced graphs restricted to the classes of P_4-tidy graphs, paw-free graphs, line graphs, and complements of line graphs.

In addition, we address the problem of recognizing balanced graphs within the studied subclasses of graphs. Perfect graphs can be recognized in polynomial time [4]. Balanced graphs can be recognized in $O((|V| + |E|)^9)$ time by means of the recognition algorithm for balanced matrices due to Zambelli [13]. Our characterizations lead to linear time recognition algorithms for balanced graphs within the classes P_4-tidy, paw-free, or line graphs, and to a $\mathrm{O}(|V|^7)$ recognition algorithm if the input graph is the complement of a line graph.

2 Characterizing and recognizing some balanced graphs

To formulate the characterizations, some further definitions are required. We say that a graph G is *F-free* if G contains no induced F. Some such graphs F are depicted in Figure 2. The *join* of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ (where $V_1 \cap V_2 = \emptyset$) is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv \mid u \in V_1, v \in V_2\})$.

A graph G is *clique-Helly* if every nonempty subfamily of pairwise intersecting cliques of G has a common vertex. The pyramids in Figure 2 are examples of graphs that are not clique-Helly. As any graph with a universal vertex is clique-Helly, a clique-Helly graph may contain any induced subgraph. Instead, a graph is *hereditary clique-Helly* [11] if all its induced subgraphs are clique-Helly. Prisner [11] showed that a graph is hereditary clique-Helly if and only if its clique-matrix contains no 3×3 submatrix with exactly two 1's per row and per column or, equivalently, if and only if it is pyramid-free. This is of interest to us, as every balanced graph is hereditary clique-Helly by [1].

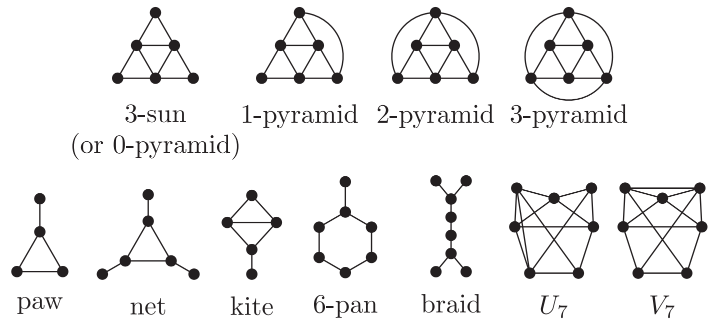

Fig. 2. Some small graphs

2.1 P_4-tidy Graphs

A graph $G = (V, E)$ is P_4*-tidy* if for every vertex set A inducing a P_4 in G there is at most one vertex $v \in V \setminus A$ such that $G[A \cup \{v\}]$ contains at least two induced P_4's. A P_4-tidy graph is perfect iff it is C_5-free (note that perfect P_4-tidy graphs are called P_4-lite and contain all P_4-free graphs, see [9]).

Theorem 2.1 *For a P_4-tidy graph G, the following statements are equivalent:*

(i) G is balanced.

(ii) G is perfect and hereditary clique-Helly.

(iii) G contains no induced C_5, 3-sun, 2-pyramid, or 3-pyramid.

This characterization and the special structure of the modular decomposition tree of P_4-tidy graphs by [9] yield an $O(|V| + |E|)$ time algorithm for deciding whether a P_4-tidy graph $G = (V, E)$ is balanced or not.

2.2 Paw-free Graphs

We now provide a minimal forbidden induced subgraph characterization of balanced graphs restricted to paw-free graphs.

Theorem 2.2 *For a paw-free graph G, the following are equivalent:*

(i) G is balanced.

(ii) G is perfect and hereditary clique-Helly.

(iii) G has no odd holes and contains no induced 3-pyramid.

(iv) Each connected component of G is either bipartite or is the join of a complete bipartite and a complete graph.

This characterization implies a linear time algorithm to decide whether a given paw-free graph G is balanced as condition (iv) can be tested in linear time.

2.3 Line Graphs

Consider a graph R, then its line graph $L(R)$ is obtained by taking one vertex for each edge of R and joining two vertices in $L(R)$ if the corresponding edges are adjacent in R. Perfect line graphs were characterized by Trotter [12]. We prove structural characterizations of those line graphs that are balanced, including a characterization by minimal forbidden induced subgraphs.

Theorem 2.3 *Let G be a line graph and let R be a graph such that $G = L(R)$. Then, the following assertions are equivalent:*

- *(i) G is balanced.*
- *(ii) G is perfect and hereditary clique-Helly.*
- *(iii) G has no odd holes and no induced 3-sun, 1-pyramid, or 3-pyramid.*
- *(iv) R has no odd cycles of length at least 5 and no partial net, kite, or K_4.*
- *(v) If U is the set of vertices of R of degree 2 whose two neighbors are adjacent and E' is the set of edges of R whose both endpoints are the neighbors of one vertex in U, then $R-U$ is bipartite and every edge of $R-U$ that is a member of E' belongs to no cycle of $R-U$.*

It is possible to derive from assertion (v) a linear time algorithm to decide in $O(|V|+|E|)$ time whether a given line graph $G=(V,E)$ is balanced or not.

2.4 Complements of Line Graphs

While the class of perfect graphs is self-complementary [10], the class of balanced graphs is not self-complementary; e.g., the net is balanced, but the 3-sun is not. We characterize those complements of line graphs that are balanced, including a characterization by minimal forbidden induced subgraphs.

Theorem 2.4 *Let G be the complement of a line graph and let R be a graph such that $G = \overline{L(R)}$. Then, the following assertions are equivalent:*

- *(i) G is balanced.*
- *(ii) A clique-matrix of G has no square submatrix of order 3, 5, or 7 with exactly two 1's per row and per column.*
- *(iii) G contains no induced 3-sun, 2-pyramid, 3-pyramid, C_5, $\overline{C_7}$, U_7, or V_7.*
- *(iv) R contains no partial bipartite claw, $P_3 \cup P_5$, $3P_3$, C_5, C_7, 6-pan, or braid.*

The theorem above yields a $O(|V|^7)$ time algorithm for deciding whether the complement $G=(V,E)$ of a line graph is balanced or not. It would be interesting to find an asymptotically faster algorithm.

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