On the Cornaz-Jost transformation to solve the graph coloring problem

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Abstract. In this paper, we use a reduction by Cornaz and Jost from the graph coloring problem to the maximum stable set problem in order to characterize new graph classes where the graph coloring problem and the more general max-coloring problem can be solved in polynomial time.

1 Introduction

A *stable set* of a graph is a subset of pairwise nonadjacent vertices, and a *coloring* of a graph is a partition of its vertices into nonempty stable sets. The maximum cardinality of a stable set of a graph G is denoted by $\alpha(G)$, and the minimum number of stable sets in a coloring of G, called the *chromatic number* of G, is denoted by $\chi(G)$. The graph coloring problem is a basic model for scheduling, frequency assignment, and resource allocation problems. From particular constraints arising in practical settings, more elaborate models of coloring have been defined in the literature.

Given a graph G with a nonnegative weight w associated to each vertex v, the *max-coloring problem* consists of finding a coloring of G that minimizes the sum, over all stable sets in the partition, of the maximum weight of a vertex in the set. It has applications in batch scheduling [1, 2] and buffer minimization [3].

The graph coloring problem is NP-complete in general, but it can be solved in polynomial time for several classes, being the most prominent the class of perfect graphs [4]. For a compendium of graph classes and the corresponding computational complexity of the coloring problem on them, see [5]. Max-coloring is substantially harder than the graph coloring problem, in particular it is NP-hard in chordal graphs [1], and so in perfect graphs.

In a recent paper, Cornaz and Jost [6] exhibit a new polynomial-time reduction from the graph coloring problem to the maximum stable set problem. Namely, given a graph G with n vertices and m edges, they construct an auxiliary graph $T(G)$ with $\bar{m}$ vertices such that the set of all stable sets of $T(G)$ is in one-to-one correspondence with the set of all colorings of G, where $\bar{m}$ is the number of edges of the complement graph $\bar{G}$ of G.

In fact, the reduction is more general and applies also to weighted graphs. They reduce the max-coloring problem to the maximum weighted stable set problem.

A *maximum weighted stable set* of G is a set of pairwise nonadjacent vertices such that the sum of the weights of the vertices in the set is maximum. The maximum weighted stable set problem and its unweighted version are NP-complete in general, and they can be solved in polynomial time for perfect graphs [4] and in $O(n^3)$ in claw-free graphs [7].

A *claw* is a graph formed by a vertex with three neighbors of degree one. A *hole* in a graph G is an induced cycle of length at least five. An *antihole* is the complement of a hole. A hole or antihole is *odd* if it has an odd number of vertices. Denote by C_k the induced cycle of length k, and by P_k the induced path of k vertices. If H is a graph, a graph G is called H*-free* if no induced subgraph of G is isomorphic to H.

A *clique* of a graph is a subset of pairwise adjacent vertices. The maximum cardinality of a clique of G is denoted by $\omega(G)$. A graph G is *perfect* when $\chi(H) = \omega(H)$ for every induced subgraph H of G. Equivalently, a graph is perfect if and only if it contains neither odd holes nor odd antiholes as induced subgraphs [8].

Given a graph G, denote by $V(G)$ and $E(G)$ the set of vertices and edges of G, respectively. Let $n = |V(G)|$ and $m = |E(G)|$. Denote by $\bar{G}$ the complement of G, and by $\bar{m}$ the number of edges of $\bar{G}$. Denote by $L(G)$ the *line graph* of G, that is, the intersection graph of the edges of G.

Let D be a simple digraph with vertex set $V(D)$ and arc set $A(D)$. An arc with tail u and head v is denoted by uv. The digraph D is called *acyclic* if it has no directed cycle. Recall that D is acyclic if and only if there is a total ordering $\prec$ on its vertex set such that $u \prec v$ for each arc uv. A pair of arcs of D is called a *simplicial pair* of D if they share the tail and their heads are connected by an arc.

Let D be an acyclic orientation of the complement $\bar{G}$ of a graph G. The graph $T(G)$ is obtained from the line graph of $\bar{G}$ by removing all edges between pairs of edges of $\bar{G}$ which are simplicial pairs of arcs in D.

Theorem 1. *[6] For any graph G and any acyclic orientation of its complement $\bar{G}$, there is a one-to-one correspondence between the set of all colorings of G and the set of all stable sets of $T(G)$. Moreover, $\alpha(T(G)) + \chi(G) = |V(G)|$.*

Note that $|V(T(G))| = \bar{m}$ and $|E(T(G))|$ is equal to the number of edges of the line graph of $\bar{G}$ minus the number of triangles in $\bar{G}$. Therefore, given a undirected graph G, the number of vertices and edges of $T(G)$ does not depend on the order of $V(G)$ from which the orientation of D is derived.

Given a graph G with a nonnegative weight w on its vertex set, denote by $\alpha_w(G)$ the weight of a maximum weighted stable set of G with respect to w, and by $\chi_w(G)$ the value of an optimum max-coloring of G with respect to w. For each vertex a of $T(G)$ corresponding to the arc uv of D, define the weight $\tilde{w}(a) := w(v)$. The previous theorem can be generalized to weighted graphs in the following way.

Theorem 2. *[6] Let G be a graph with a nonnegative weight w on its vertices, and consider an acyclic orientation of $\bar{G}$ given by a non-increasing ordering of $V(G)$ with respect to w. Then $\alpha_{\tilde{w}}(T(G)) + \chi_w(G) = w(V(G))$.*

In this paper, we analyze the transformation T. In particular, we try to find classes of graphs $\mathcal{C}$ such that $T(\mathcal{C})$ is a class of graphs where the maximum weighted stable set is polynomial-time solvable. In general, we consider $T(\mathcal{C})$ as the class of graphs $T(G)$ obtained from any acyclic orientation of the complement of a graph G in $\mathcal{C}$ since, for the max-coloring problem, this orientation is given by the weight function. But one of our main results is the characterization of a class of graphs $\mathcal{C}$ in which, for every graph G in $\mathcal{C}$, there exists an orientation of $\bar{G}$ such that $T(G)$ is claw-free. In that class, the coloring problem is polynomial-time solvable, by using any available polynomial-time algorithm for maximum stable set in claw-free graphs. The class obtained is not contained in any previously known class where the coloring problem is polynomial-time solvable, as far as we could check in [5]. Moreover, it can be recognized in polynomial time, which makes the result interesting also from a practical point of view.

2 Main results

We start by analyzing the pre-image of perfect graphs by the transformation T.

Proposition 1. *If G has an odd hole as induced subgraph then $T(G)$ has a hole of length 5 as induced subgraph, and if G has an odd antihole $\overline{C_k}$ as induced subgraph then $T(G)$ has an odd hole C_k as induced subgraph.*

Corollary 1. *If $T(G)$ is a perfect graph then G is a perfect graph. That is, T^{-1}(perfect graphs) $\subseteq$ perfect graphs.*

Note that if G is perfect then $T(G)$ is not necessarily perfect. In particular, there exists a perfect graph G such that $T(G)$ is not perfect, independently of the acyclic orientation of its complement. So, from the graph coloring point of view, the pre-image of perfect graphs by the transformation T leads to a class of graphs where the problem is already known to be polynomial-time solvable. But it could be of interest to characterize T^{-1}(perfect graphs) from the max-coloring point of view.

Our aim now is to characterize T^{-1}(claw-free graphs), in order to describe a new class in which the max-coloring problem can be solved in polynomial time.

Theorem 3. *$T(G)$ is a claw-free graph for every acyclic orientation of $\bar{G}$ if and only if G does not contain P_5 or a graph in Fig. 1 as an induced subgraph.*

Corollary 2. *Given a graph G that does not contain P_5 or a graph in Fig. 1 as an induced subgraph, and a nonnegative weight on its vertices, the max-coloring problem on G can be solved in $O(\bar{m}^3)$.*

We can strength this result for coloring by choosing a clever ordering of the vertices of G in order to obtain an orientation of the graph $\bar{G}$ leading to a claw-free graph $T(G)$, even if we allow to have some induced P_5's in G.

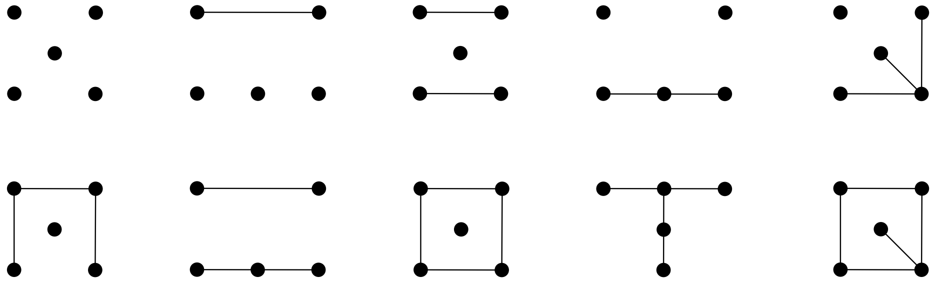

Fig. 1. Some of the forbidden induced subgraphs for a graph G such that every acyclic orientation of $\bar{G}$ gives rise to a claw-free graph $T(G)$.

Theorem 4. *If G does not contain a graph in Fig. 1 as an induced subgraph and there exists an ordering $\prec$ of the vertices of G such that for every induced $P_5 = v_1v_2v_3v_4v_5$ of G it holds $v_3 \prec v_1$ and $v_3 \prec v_5$, then the graph $T(G)$, obtained from the acyclic orientation of $\bar{G}$ given by that ordering, is a claw-free graph.*

From Theorem 4, one can characterize the following class of graphs in which the coloring problem can be solved in polynomial time.

Corollary 3. *Given a graph G, it can be checked in polynomial time if G does not contain a graph in Fig. 1 as an induced subgraph and there exists an ordering $\prec$ of the vertices of G such that for every induced $P_5 = v_1v_2v_3v_4v_5$ of G it holds $v_3 \prec v_1$ and $v_3 \prec v_5$. In this case, the coloring problem can be solved in polynomial time for G. The overall complexity of the algorithm is $O(n^5 + \bar{m}^3)$.*

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Appendix: Figures and proofs

We denote with uv the vertex in $V(T(G))$ corresponding to the arc uv of D.

Proof of Proposition 1. If G has $\overline{C_k}$ as induced subgraph, then $\bar{G}$ has C_k as induced subgraph, and no pair of edges of it is a simplicial pair of arcs in D. So $T(G)$ has C_k as induced subgraph. In particular, for $k = 5$, if G has $\overline{C_5} = C_5$ as induced subgraph, then $T(G)$ has C_5 as induced subgraph.

Let H be a subgraph of G inducing an odd hole with vertex set $V(H) = \{v_1, \ldots, v_k\}$, where $k \geq 7$ and v_1 is the smallest vertex of H in the vertex ordering defining the orientation of D. Then, vertices $\{v_1v_5, v_1v_4, v_4v_6, v_3v_6, v_3v_5\}$ induce a hole of length 5 in $T(G)$. □

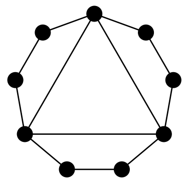

Fig. 2. The complement $\bar{G}$ of a perfect graph G such that $T(G)$ is not perfect, independently of the orientation of $\bar{G}$.

Theorem 3 follows from this two propositions.

Proposition 2. *If $T(G)$ is a claw-free graph then $\alpha(G) \leq 4$.*

Proof. Proposition 2 Suppose that $\alpha(G) \geq 5$. Then $\bar{G}$ contains a clique of size 5, namely $\{v_1, v_2, v_3, v_4, v_5\}$. Suppose without loss of generality that, in the vertex ordering of $\bar{G}$ inducing the acyclic orientation of D, it holds $v_1 \prec v_2 \prec v_3 \prec v_4 \prec v_5$. Then, vertices $\{v_1v_2, v_2v_3, v_2v_4, v_2v_5\}$ induce a claw in $T(G)$. □

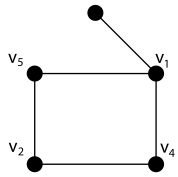

Fig. 3. G'.

Proposition 3. *$T(G)$ is a claw-free graph for every acyclic orientation of $\bar{G}$ if and only if G does not contain a spanning subgraph of G' (Fig. 3) as an induced subgraph.*

Proof. Suppose first that G contains a spanning subgraph H of G' as induced subgraph. Let us consider an ordering of the vertices of $\bar{G}$ such that $v_1 \prec v_2 \prec v_3 \prec v_4 \prec v_5$, and let D be the digraph obtained from that ordering. Then v_3v_4 and v_3v_5 are a simplicial pair of D, so they are nonadjacent in $T(G)$. Besides, v_2v_3 does not form a simplicial pair of D with neither v_1v_2, v_3v_4 nor v_3v_5, independently of which edges were removed from G' to obtain H. So, $\{v_2v_3, v_1v_2, v_3v_4, v_3v_5\}$ induces a claw in $T(G)$.
Conversely, suppose that $T(G)$ contains an induced claw, for some ordering of the vertices of $\bar{G}$. Let ab, cd, ef and gh be the edges of $\bar{G}$ inducing the claw on $T(G)$, where cd, ef and gh form a stable set on $T(G)$ and ab is adjacent to all of them. We will split the proof into two cases: cd, ef and gh share the same endpoint of ab (wlog, $c = e = g = a$), or two of them share an endpoint with ab and the third one shares the other endpoint with ab (wlog, $c = e = a$ and $g = b$). In the first case, each pair of edges in $\{cd, ef, gh\}$ is a simplicial pair, so d, f and h form a triangle in $\bar{G}$, and $\{b, a, d, f, h\}$ induces on G a subgraph of G'. In the second case, $\{cd, ef\}$ is a simplicial pair, so d and f are adjacent in $\bar{G}$, and $a \prec d, f$. If h is different from d and f, then $\{h, b, a, d, f\}$ induces on G a subgraph of G'. So, suppose wlog that $h = d$. In that case, $gh(= bd)$ and $cd(= ad)$ should form a simplicial pair, but this is impossible because $a \prec d$, and this completes the proof of the proposition. □

Proof of Theorem 4. Let G be a graph and consider an ordering $\prec$ of its vertices. Suppose, by the way of contradiction, that $T(G)$ is not a claw-free graph. Then, by Proposition 3, G contains either a graph in Fig. 1 or a P_5 $H = v_1v_2v_3v_4v_5$ as induced subgraph. Since G does not contain a graph in Fig. 1 as an induced subgraph, then the claw in $T(G)$ is formed by the edges of $\bar{H}$. It is easy to check that they induce a claw in $T(G)$ if and only if the either $v_1 \prec v_3$ or $v_5 \prec v_3$. □

Proof of Corollary 3. It can be checked in $O(n^5)$ time that G does not contain a graph in Fig. 1 as an induced subgraph. Also in $O(n^5)$ time it can be builded a digraph D' with vertex set $V(G)$ and an oriented arc for each pair of vertices x, y such that there is an induced P_5 in G such that x is the middle vertex and y is an end vertex of it. Finally, it can be checked in linear time if D' is acyclic, and in that case it can be given in linear time a suitable ordering for $V(G)$. Finally, the algorithm to solve the coloring problem consists on building the graph $T(G)$ with respect to that order and, by Theorem 1, solving the stable set problem on it. By Theorem 4, $T(G)$ is a claw-free graph and so, the stable set problem can be solved in $O(\bar{m}^3)$ time [7]. □