Minimum Sum Coloring of P_4-sparse graphs [1]

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Abstract

In this paper, we study the Minimum Sum Coloring (MSC) problem on P_4-sparse graphs. In the MSC problem, we aim to assign natural numbers to vertices of a graph such that adjacent vertices get different numbers, and the sum of the numbers assigned to the vertices is minimum. First, we introduce the concept of maximal sequence associated with an optimal solution of the MSC problem of any graph. Next, based in such maximal sequences, we show that there is a large sub-family of P_4-sparse graphs for which the MSC problem can be solved in polynomial-time.

Keywords: graph coloring, minimum sum coloring, P_4-sparse graphs.

1 Introduction

In this paper, we study the *Minimum Sum Coloring (MSC)* problem for the family of P_4-sparse graphs. A *vertex coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices in V such that adjacent vertices receive

[1] Partially supported by ANPCyT PICT-2007-00518 and 00533, UBACyT Grants X069 and X606 (Arg.), nd BQR-UPN-2008 Grant (France).

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different colors. We assume that the colors are positive integers. A vertex k-coloring of a graph G is a coloring where the color of each vertex in V is taken from the set $\{1, 2, \ldots, k\}$. Given a vertex coloring of a graph G, the *sum* of the coloring is the sum of the colors assigned to the vertices. The *chromatic sum* $\Sigma(G)$ of G is the smallest sum that can be achieved by any proper coloring of G. In the *Minimum Sum Coloring* (MSC) problem we have to find a coloring of G with sum $\Sigma(G)$. The minimum number of colors needed in a minimum sum coloring of G is called the *strength* of G and it is denoted by $s(G)$. Clearly, for any graph G we have that $s(G) \geq \chi(G)$, where $\chi(G)$ denote the chromatic number of G.

The MSC problem was introduced by Kubicka [10]. The problem is motivated by applications in scheduling [1,2,5] and VLSI design [12,14]. The computational complexity of determining the vertex chromatic sum of a simple graph has been studied extensively since then. In [11] it is shown that the problem is NP-hard in general, but polynomial-time solvable for trees. The dynamic programming algorithm for trees can be extended to partial k-trees and block graphs [9]. Furthermore, the MSC problem is NP-hard even when restricted to some classes of graphs for which finding the chromatic number is easy, such as bipartite or interval graphs [2,14]. A number of approximability results for various classes of graphs were obtained in the last ten years [1,4,5,3].

Jansen has shown in [9] that a more general optimization problem where each color has an integer cost, but this cost is not necessarily equal to the color itself, *the Optimal Cost Chromatic Partition (OCCP) problem*, can be solved in polynomial-time for cographs and block graphs, but it remains NP-hard for permutation graphs. Salavatipour has shown in [13] that the OCCP problem can be solved in polynomial-time for the family of P_4-reducible graphs, a superclass containing the family of cographs. P_4-sparse graphs were introduced in [6]. They generalize cographs and P_4-reducible graphs, can be recognized in linear time [7], and are a subclass of perfect graphs [6].

A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. Clearly, the complement of a thin spider is a thick spider, and vice-versa. The triple (S, C, R) is called the *spider partition*, and can be found in linear time [7]. P_4-sparse graphs have a nice decomposition theorem as follows.

Theorem 1.1 [6,8] *If G is a non-trivial P_4-sparse graph, then either G or $\overline{G}$ is not connected, or G is a spider.*

To each P_4-sparse graph G one can associate a corresponding decomposition rooted tree T in the following way. Each non-leaf node in the tree is labeled with either "$\cup$" (union-nodes), or "$\vee$" (join-nodes) or "SP" (spider-partition-nodes), and each leaf is labeled with a vertex of G. Each non-leaf node has two or more children. Let T_x be the subtree of T rooted at node x and let V_x be the set of vertices corresponding to the leaves in T_x. Then, each node x of the tree corresponds to the graph $G_x = (V_x, E_x)$. An union-node (join-node) corresponds to the disjoint union (join) of the P_4-sparse graphs associated with the children of the node. A spider-partition-node corresponds to a spider-partition (S, C, R) of the P_4-sparse graphs associated with the children of the node. Finally, the P_4-sparse graph that is associated with the root of the tree is just G, the P_4-sparse graph represented by this decomposition tree. The decomposition tree associated to a P_4-sparse graph can be computed in linear time [8].

2 Maximal sequences and optimal solutions of the MSC problem

A k-coloring of a graph $G = (V, E)$ is a partition of the vertex set V into k independent sets $S_1, \ldots, S_k$, where each vertex in S_i is colored with color i, for $1 \leq i \leq k$. So, for any such k-partition of V into independent sets, we can associate a non-negative *sequence* p such that $p[i] = |S_i|$ for $i = 1, \ldots, k$ and $p[i] = 0$ for $i > k$. In the sequel, we deal only with finite support non-negative integer sequences. Let $|p| = \max\{i : p[i] > 0\}$.

Definition 2.1 Let p and q be two integer sequences. We say that p dominates q, denoted by $p \succeq q$, if for all $t \geq 1$ holds that $\sum_{1 \leq i \leq t} p[i] \geq \sum_{1 \leq i \leq t} q[i]$.

The following two lemmas are direct consequences of Definition 2.1.

Lemma 2.2 *The dominance relation $\succeq$ is a partial order.*

Lemma 2.3 *Let p be a sequence and let $\widetilde{p}$ be the sequence that results from p when we order it in a non-decreasing way. Then, $\widetilde{p} \succeq p$.*

The following lemma will be very useful in order to study the sum-coloring problem on graphs. All the proofs are omitted due to space constraints.

Lemma 2.4 *Let p and q be two sequences and let $n = \max\{|p|, |q|\}$. If $p \succeq q$ and $\sum_{1 \leq i \leq n} p[i] = \sum_{1 \leq i \leq n} q[i]$, then it holds that $\sum_{1 \leq i \leq n} i.p[i] \leq \sum_{1 \leq i \leq n} i.q[i]$.*

Notice that if the sequences represent partitions of the vertex set of a graph into independent sets, where the value of the ith element of the sequence represents the size of the ith independent set in the partition, then for the sum-coloring problem on graphs we can restrict us to study maximal sequences w.r.t. the partial order $\succeq$. Notice also that maximal sequences are non-increasing sequences. In the following, we define some operations between sequences.

Definition 2.5 Let p and q be two sequences. The join of p and q, denoted by $p \star q$, is the sequence that results of ordering in a non-increasing way the concatenation of sequences p and q.

Definition 2.6 Let p and q be two sequences. The sum of p and q, denoted by $p + q$, is the sequence such that its i-th value is equal to $p[i] + q[i]$, for $i = 1, \ldots, \max\{|p|, |q|\}$.

Definition 2.7 Let p and q be two sequences. We say that p and q are non-comparable, denoted by $p||q$, if $p \not\succeq q$ and $q \not\succeq p$.

The following two lemmas will be useful in order to study the MSC problem on P_4-sparse graphs.

Lemma 2.8 *Let p, p' and q be sequences. If $\widetilde{p} \succeq \widetilde{p}'$ then $p \star q \succeq p' \star q$.*

Lemma 2.9 *Let p, p' and q be sequences. Then, $p||p'$ if and only if $p+q||p'+q$.*

A direct consequence of previous lemma is the following result.

Corollary 2.10 *Let p, p' and q be sequences. Then, $p \succeq p'$ if and only if $p + q \succeq p' + q$.*

Let G be a graph. Suppose that $G = G_1 \cup G_2$ and let p be a sequence representing a partition of the vertex set of G into independent sets. Clearly, $p = p_1 + p_2$ where p_1 (resp. p_2) is a sequence representing a partition of the vertex set of G_1 (resp. G_2) into independent sets. In an analogous way, suppose that $G = G_1 + G_2$ and let p be a sequence representing a partition of the vertex set of G into independent sets. Clearly, $p = p_1 \star p_2$ where p_1 (resp. p_2) is a sequence representing a partition of the vertex set of G_1 (resp. G_2) into independent sets. Therefore, by Corollary 2.10 and Lemma 2.8, if we are looking for maximal sequences of G representing partitions of its vertex set into independent sets then, in both cases it is sufficient to consider maximal partitions of the graphs G_1 and G_2.

3 Maximal sequences of P_4-sparse graphs

We will present here the main results of this work. In the sequel, sequences of a graph will represent partitions of its vertex set into independent sets.

Lemma 3.1 *Let $G = (S, C, R)$ be a thin spider. Then,*

(i) *If $R = \emptyset$ then, G has only one maximal sequence p, with $|p| = |C|$, where $p[1] = |C|$, $p[2] = 2$, and $p[i] = 1$ for $3 \leq i \leq |C|$.*

(ii) *If $R \neq \emptyset$ then, the number of maximal sequences of G is equal to the number of maximal sequences of $G[R]$. Moreover, for each maximal sequence q of $G[R]$ there exists only one maximal partition q' of G with $|q'| = |q| + |C|$ and where $q'[1] = q[1] + |C|$, $q'[i] = q[i]$ for $2 \leq i \leq |q|$, and $q'[i] = 1$ for $|q| + 1 \leq i \leq |q| + |C|$.*

Lemma 3.2 *Let $G = (S, C, R)$ be a thick spider. Then,*

(i) *If $R = \emptyset$ then, G has only two maximal sequences p_1 and p_2, with $|p_1| = |C|$ and $|p_2| = |C| + 1$, where $p_1[i] = 2$ for $1 \leq i \leq |C|$, and $p_2[1] = |C|$ and $p_2[i] = 1$ for $2 \leq i \leq |C| + 1$.*

(ii) *If $R \neq \emptyset$ then, the number of maximal sequences of G is equal to the number of maximal sequences of $G[R]$. Moreover, for each maximal sequence q of $G[R]$ there exists only one maximal partition q' of G with $|q'| = |q| + |C|$ and where $q'[1] = q[1] + |C|$, $q'[i] = q[i]$ for $2 \leq i \leq |q|$, and $q'[i] = 1$ for $|q| + 1 \leq i \leq |q| + |C|$.*

Notice also that the trivial graph has only one maximal sequence p, with $|p| = 1$, where $p[1] = 1$. Therefore, we have the following theorems.

Theorem 3.3 *Let G be a P_4-sparse graph such that in its modular decomposition there are no thick spiders (S, C, R) with $R = \emptyset$. Then,*

(i) *$s(G) = \chi(G)$, and $\Sigma(G)$ and an optimal coloring of G can be computed from its modular decomposition in polynomial time.*

(ii) *In such an optimal coloring, each independent set S_i is a maximum independent set of $G \setminus \bigcup_{1 \leq j < i} S_j$ which verifies $\chi(G \setminus \bigcup_{1 \leq j \leq i} S_j) = \chi(G \setminus \bigcup_{1 \leq j < i} S_j) - 1$.*

Theorem 3.4 *Let G be a P_4-sparse graph on n vertices. Let k be the number of thick spiders (S, C, R) with $R = \emptyset$ in the modular decomposition of G. Then, $s(G) \leq \chi(G) + k$, the number of maximal sequences of G is at most 2^k, and an optimal coloring of G can be computed in $2^k P(n)$ time, where $P(n)$ is a polynomial on n.*

References

[1] A. Bar-Noy, M. Bellare, M. M. Halldórsson, H. Shachnai, and T. Tamir. On chromatic sums and distributed resource allocation. *Information and Computation*, 140(2):183–202, 1998.

[2] A. Bar-Noy and G. Kortsarz. Minimum color sum of bipartite graphs. *J. Algorithms*, 28(2):339–365, 1998.

[3] U. Feige, L. Lovász, and P. Tetali. Approximating min sum set cover. *Algorithmica*, 40(4):219–234, 2004.

[4] K. Giaro, R. Janczewski, M. Kubale, and M. Malafiejski. Approximation algorithm for the chromatic sum coloring of bipartite graphs. In *Proc. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, volume 2462 of *Lecture Notes in Computer Science*, pages 135–145. Springer, 2002.

[5] M. M. Halldórsson, G. Kortsarz, and H. Shachnai. Sum coloring interval and k-claw free graphs with application to scheduling dependent jobs. *Algorithmica*, 37(3):187–209, 2003.

[6] C. T. Hoàng. Perfect graphs. Ph.D. Thesis, School of Computer Science, McGill University, 1985.

[7] B. Jamison and S. Olariu. Recognizing P_4-sparse graphs in linear time. *SIAM J. on Comput.*, 21:381–406, 1992.

[8] B. Jamison and S. Olariu. A tree representation for P_4-sparse graphs. *Disc. Appl. Math.*, 35:115–129, 1992.

[9] K. Jansen. Complexity results for the optimum cost chromatic partition problem. In *Proc. of ICALP 1997*, volume 1256 of *Lecture Notes in Computer Science*, pages 727–737. Springer-Verlag, 1997.

[10] E. Kubicka. The Chromatic Sum of a Graph. PhD thesis, Western Michigan University, 1989.

[11] E. Kubicka and A. J. Schwenk. An introduction to chromatic sums. In *Proceedings of the ACM Computer Science Conf.*, pages 15–21. Springer, 1989.

[12] S. Nicoloso, M. Sarrafzadeh, and X. Song. On the sum coloring problem on interval graphs. *Algorithmica*, 23(2):109–126, 1999.

[13] M. Salavatipour. On sum coloring of graphs. Master Thesis, Department of Computer Science, University of Toronto, 2000.

[14] T. Szkaliczki. Routing with minimum wire length in the dogleg-free manhattan model is NP-complete. *SIAM J. Comput.*, 29(1):274–287, 1999.

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