

Partial characterizations of clique-perfect graphs II: diamond-free and Helly circular-arc graphs

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Abstract

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G . A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. A graph G is *clique-perfect* if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of G . The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. Recently we were able to characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph belongs to two different subclasses of claw-free graphs. These characterizations lead to polynomial time recognition of clique-perfect graphs in these classes of graphs. In this paper we solve the characterization problem in two new classes of graphs: diamond-free and Helly circular-arc (*HCA*) graphs. This last characterization leads to a polynomial time recognition algorithm for clique-perfect *HCA* graphs.

Key words: Clique-perfect graphs, diamond-free graphs, Helly circular-arc graphs, K-perfect graphs, perfect graphs.

1 Introduction

Let G be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by \overline{G} , the complement of G . Given two graphs G and G' we say that G *contains* G' if G' is isomorphic to an induced subgraph of G . When we need to refer to the non-induced subgraph containment relation, we will say so explicitly.

A class of graphs \mathcal{C} is *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to \mathcal{C} .

The neighborhood of a vertex v is the set $N(v)$ consisting of all the vertices which are adjacent to v . The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v of G is *universal* if $N[v] = V(G)$. Two vertices v and w are *twins* if $N[v] = N[w]$; and u *dominates* v if $N[v] \subseteq N[u]$. For an induced subgraph H of G and a vertex v in $V(G) \setminus V(H)$, the *set of neighbors of v in H* is the set $N(v) \cap V(H)$.

A *complete set* or just a *complete* of G is a subset of vertices pairwise adjacent. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G .

A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being adjacent if and only if the corresponding sets intersect.

A *circular-arc graph* is the intersection graph of arcs of the unit circle. A *representation* of a circular-arc graph is a collection of arcs (of the unit circle), each corresponding to a unique vertex of the graph, such that two intervals intersect if and only if the corresponding vertices are adjacent. A *Helly circular-arc (HCA) graph* is a circular-arc graph admitting a representation whose arcs

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satisfy the Helly property. In particular, in a Helly circular-arc representation of a graph, for every clique there is a point of the circle that belongs to the arcs corresponding to the vertices in the clique, and to no others. We call such a point an *anchor* of the clique (please note that an anchor may not be unique).

A graph is *clique-Helly* (*CH*) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (*HCH*) if H is clique-Helly for every induced subgraph H of G .

Let G be a graph and H a subgraph of G (not necessarily induced). The graph H is a *clique subgraph* of G if every clique of H is a clique of G (we use the definition of [8]).

A complete graph on n vertices is denoted by K_n . A *diamond* is the graph isomorphic to $K_4 \setminus \{e\}$, where e is an edge of K_4 . A graph is *diamond-free* if it does not contain a diamond.

A *claw* is the graph isomorphic to $K_{1,3}$. A graph is *claw-free* if it does not contain a claw. The *line graph* $L(G)$ of G is the intersection graph of the edges of G . A graph F is a *line graph* if there exists a graph H such that $L(H) = F$. Clearly, line graphs are a subclass of claw-free graphs.

A *hole* is a chordless cycle of length $n \geq 4$, and it is denoted by C_n . An *antihole* is the complement of a hole. A hole or antihole on n vertices is said to be *odd* if n is odd. A *4-wheel* is a graph on five vertices v_1, \dots, v_5 , such that $v_1v_2v_3v_4v_1$ is a hole and v_5 is adjacent to all of v_1, v_2, v_3, v_4 .

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G . The *clique covering number* of G , denoted by $k(G)$, is the cardinality of a minimum clique cover of G . An obvious lower bound is the maximum cardinality of the stable sets of G , the *stability number* of G , denoted by $\alpha(G)$. A graph G is *perfect* if $\alpha(H) = k(H)$ for every induced subgraph H of G . It has been proved recently that a graph G is perfect if and only if no induced subgraph of G is an odd hole or an odd antihole [14], and that perfect graphs can be recognized in polynomial time [13].

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G . A graph G is *K-perfect* if $K(G)$ is perfect.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G . A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G , denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G , respectively. It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph G . A graph G is *clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G . Say that a graph is *clique-imperfect* when

it is not clique-perfect. Clique-perfect graphs have been implicitly studied quite extensively since the earliest results by Berge [1,3,4,9,11,18,22,23], but the terminology “clique-perfect” was introduced in [21]. Some known classes of clique-perfect graphs are dually chordal graphs [9], comparability graphs [1], balanced graphs [4] and distance-hereditary graphs [22].

The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem.

There are some partial results in this direction. In [23], clique-perfect graphs are characterized by minimal forbidden subgraphs for the class of chordal graphs. In [24], minimal graphs G with $\alpha_c(G) = 1$ and $\tau_c(G) > 1$ are explicitly described. In [6], clique-perfect graphs are characterized by minimal forbidden subgraphs for two subclasses of claw-free graphs. These characterizations lead to polynomial algorithms for recognizing clique-perfect graphs in these subclasses.

In this paper, we characterize diamond-free clique-perfect graphs by a list of non minimal forbidden subgraphs. Moreover, we give a characterization of clique-perfect graphs for the whole class of Helly circular-arc graphs by minimal forbidden subgraphs. As a corollary of this characterization we can find a polynomial time recognition algorithm for clique-perfect *HCA* graphs.

Definitions not given here appear in [20,10]. An interesting survey on graph classes can be found in [27]. Preliminary results of this paper were published in [5,7].

2 New families and partial characterizations

In this section we introduce various families of clique-imperfect graphs, needed to characterize diamond-free and *HCA* clique-perfect graphs by forbidden subgraphs.

A *sun* (or trampoline) is a chordal graph G on $2r$ vertices whose vertex set can be partitioned into two sets, $W = \{w_1, \dots, w_r\}$ and $U = \{u_1, \dots, u_r\}$, such that W is a stable set and for each i and j , w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$. A sun is *odd* if r is odd. A sun is *complete* if U is a complete.

A *generalized sun* is defined as follows. Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* (or *improper*) if it forms a triangle with some vertex of C . An *r-generalized sun*, $r \geq 3$, is a graph

G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non proper edges $\{e_j\}_{j \in J}$ (J is permitted be an empty set) and a stable set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, u_j is adjacent only to the endpoints of e_j . An r -generalized sun is said to be *odd* if r is odd. Clearly odd holes and odd suns are odd generalized suns.

Theorem 1 [8] *Odd generalized suns and antiholes of length $t = 1, 2 \pmod 3$ ($t \geq 5$) are not clique-perfect.*

Unfortunately, not every odd generalized sun is minimally clique-imperfect (with respect to taking induced subgraphs). Nevertheless, odd holes and complete odd suns are minimally clique-imperfect, and we will distinguish other two kinds of minimally clique-imperfect odd generalized suns in order to state a characterization of *HCA* clique-perfect graphs by minimal forbidden induced subgraphs.

A *viking* is a graph G such that $V(G) = \{a_1, \dots, a_{2k+1}, b_1, b_2\}$, $k \geq 2$, $a_1 \dots a_{2k+1}a_1$ is a cycle with only one chord a_2a_4 ; b_1 is adjacent to a_2 and a_3 ; b_2 is adjacent to a_3 and a_4 , and there are no other edges in G .

A *2-viking* is a graph G such that $V(G) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3\}$, $k \geq 2$, $a_1 \dots a_{2k+1}a_1$ is a cycle with only two chords, a_2a_4 and a_3a_5 ; b_1 is adjacent to a_2 and a_3 ; b_2 is adjacent to a_3 and a_4 ; b_3 is adjacent to a_4 and a_5 , and there are no other edges in G .

Proposition 2 *Vikings and 2-vikings are clique-imperfect.*

PROOF. They are odd generalized suns, where in both cases the odd cycle is $a_1 \dots a_{2k+1}a_1$, and the stable sets are $\{b_1, b_2\}$ and $\{b_1, b_2, b_3\}$, respectively. \square

Next we introduce two new families (which are not odd generalized suns or antiholes) of minimal clique-imperfect graphs.

Define the graph S_k , $k \geq 2$, as follows: $V(S_k) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3\}$, $a_1 \dots a_{2k+1}a_1$ is a cycle with only one chord a_3a_5 ; b_1 is adjacent to a_1 and a_2 ; b_2 is adjacent to a_4 and a_5 ; b_3 is adjacent to a_1, a_2, a_3 and a_4 , and there are no other edges in S_k .

Define the graph T_k , $k \geq 2$, as follows: $V(T_k) = \{a_1, \dots, a_{2k+1}, b_1, \dots, b_5\}$, $a_1 \dots a_{2k+1}a_1$ is a cycle with only two chords, a_2a_4 and a_3a_5 ; b_1 is adjacent to a_1 and a_2 ; b_2 is adjacent to a_1, a_2 and a_3 ; b_3 is adjacent to a_1, a_2, a_3, a_4, b_2 and b_4 ; b_4 is adjacent to a_3, a_4 and a_5 ; b_5 is adjacent to a_4 and a_5 , and there are no other edges in T_k .

Proposition 3 *Let $k \geq 2$. Then S_k and T_k are clique-imperfect.*

PROOF. Every clique of S_k contains at least two vertices of a_1, \dots, a_{2k+1} , so $\alpha_c(S_k) \leq k$. The same holds for T_k , so $\alpha_c(T_k) \leq k$. On the other hand, consider in S_k the family of cliques $\{a_1, a_2, b_1\}$, $\{a_2, a_3, b_3\}$, $\{a_3, a_4, b_3\}$, $\{a_4, a_5, b_2\}$ and either $\{a_5, a_1\}$, if $k = 2$, or $\{a_5, a_6\}, \dots, \{a_{2k+1}, a_1\}$, if $k > 2$. No vertex of S_k belongs to more than two of these $2k+1$ cliques, so $\tau_c(S_k) \geq k+1$. Analogously, consider in T_k the family of cliques $\{a_1, a_2, b_1\}$, $\{a_2, a_3, b_2, b_3\}$, $\{a_3, a_4, b_3, b_4\}$, $\{a_4, a_5, b_5\}$ and either $\{a_5, a_1\}$, if $k = 2$, or $\{a_5, a_6\}, \dots, \{a_{2k+1}, a_1\}$, if $k > 2$. No vertex of T_k belongs to more than two of these $2k+1$ cliques, so $\tau_c(T_k) \geq k+1$. \square

The minimality of vikings, 2-vikings, S_k and T_k ($k \geq 2$) will be proved as a corollary of the main theorem of Section 4.

In [24] the minimal graphs G such that $K(G)$ is complete (i.e. $\alpha_c(G) = 1$) and no vertex of G is universal (i.e. $\tau_c(G) > 1$) are characterized. The graph Q_n , $n \geq 3$, is defined as follows: $V(Q_n) = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ is a set of $2n$ vertices; v_1, \dots, v_n induce $\overline{C_n}$; for each $1 \leq i \leq n$, $N[u_i] = V(Q_n) - \{v_i\}$.

The following result will be useful to us:

Theorem 4 [24] *For $k \geq 1$, $\alpha_c(Q_{2k+1}) = 1$ and $\tau_c(Q_{2k+1}) = 2$. Moreover, if G is a graph such that $\alpha_c(G) = 1$ and $\tau_c(G) > 1$, then G contains Q_{2k+1} for some $k \geq 1$.*

For some classes of graphs, it is enough to exclude some odd generalized suns and some antiholes in order to guarantee that the graph is clique-perfect:

Theorem 5 [23] *Let G be a chordal graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd sun.*

Theorem 6 [6] *Let G be a line graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd hole or a 3-sun.*

Theorem 7 [6] *Let G be an HCH claw-free graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd hole or an antihole of length seven.*

A similar result holds for diamond-free graphs. This, however, is not the case for *HCA* graphs. The graphs S_k and T_k are minimal clique-imperfect *HCA* graphs; but these are the only minimal clique-imperfect *HCA* graphs which are not odd generalized suns or antiholes.

Our main results are the following two theorems:

Theorem 8 *Let G be a diamond-free graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd generalized sun.*

Theorem 9 *Let G be an HCA graph. Then G is clique-perfect if and only if it does not contain a 3-sun, an antihole of length seven, an odd hole, a viking, a 2-viking or one of the graphs S_k or T_k .*

In the next two sections we prove Theorems 8 and 9.

3 Diamond-free graphs

The following lemma establishes a connection between the parameters involved in the definition of clique-perfect graphs and those corresponding to perfect graphs.

Lemma 10 [8] *Let G be a graph. Then:*

- (1) $\alpha_c(G) = \alpha(K(G))$.
- (2) $\tau_c(G) \geq k(K(G))$. Moreover, if G is clique-Helly, then $\tau_c(G) = k(K(G))$.

Hereditary clique-Helly graphs are of particular interest because in this case it follows from Lemma 10 that if $K(H)$ is perfect for every induced subgraph H of G , then G is clique-perfect (the converse is not necessarily true).

The class of hereditary clique-Helly graphs can be characterized by forbidden induced subgraphs.

Theorem 11 [25,26] *A graph G is hereditary clique-Helly if and only if it does not contain the graphs of Figure 1.*

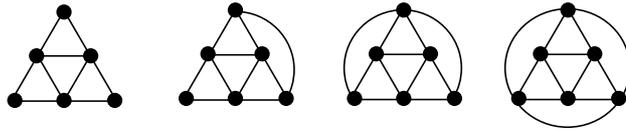


Fig. 1. Forbidden induced subgraphs for hereditary clique-Helly graphs: (left to right) 3-sun (or 0-pyramid), 1-pyramid, 2-pyramid and 3-pyramid.

As a direct corollary of this characterization, it follows that diamond-free graphs are *HCH*.

The following is a useful fact about hereditary clique-Helly graphs:

Proposition 12 *Let \mathcal{L} be a hereditary graph class, which is HCH and such that every graph in \mathcal{L} is K -perfect. Then every graph in \mathcal{L} is clique-perfect.*

PROOF. Let G be a graph in \mathcal{L} . Let H be an induced subgraph of G . Since \mathcal{L} is hereditary, H is a graph in \mathcal{L} , so it is K -perfect. Since \mathcal{L} is an HCH class, H is clique-Helly and then, by Lemma 10, $\alpha_C(H) = \alpha(K(H)) = k(K(H)) = \tau_C(H)$, and the result follows. \square

We can now prove the main result of this section.

Proof of Theorem 8. By Theorem 1, if G is clique-perfect then no induced subgraph of G is an odd generalized sun. Let us prove the converse. Let G be a diamond-free graph such that no induced subgraph of G is an odd generalized sun.

First we show that $K(G)$ contains no odd holes or odd antiholes, and therefore it is perfect. By [12], G being diamond-free implies that $K(G)$ is diamond-free, and hence $K(G)$ contains no antihole of length at least 7. Suppose $K(G)$ contains an odd hole $k_1 k_2 \dots k_{2n+1}$, where k_1, \dots, k_{2n+1} are cliques of G . Then G contains an odd cycle $v_1 v_2 \dots v_{2n+1} v_1$, where v_i belongs to $k_i \cap k_{i+1}$ and no other k_j . Since G contains no odd generalized suns, we may assume that some edge of this cycle, say, $v_1 v_2$ is in a triangle with another vertex of the cycle, say v_m . Now v_1, v_2 both belong to k_2 , and v_m does not. Since k_2 is a clique, it follows that v_m has a non-neighbor w in k_2 . But now $\{v_1, v_2, v_m, w\}$ induces a diamond, a contradiction. Finally, Proposition 12 completes the proof. \square

We remark that this characterization does not immediately lead to a polynomial time recognition algorithm for diamond-free clique-perfect graphs. The complexity of recognition of odd generalized suns is still unknown, even for diamond-free graphs.

4 Helly circular-arc graphs

The main result of this section is the following: if a graph G is HCA , then G is clique-perfect if and only if it does not contain the graphs of Figure 2. (This is Theorem 9.)

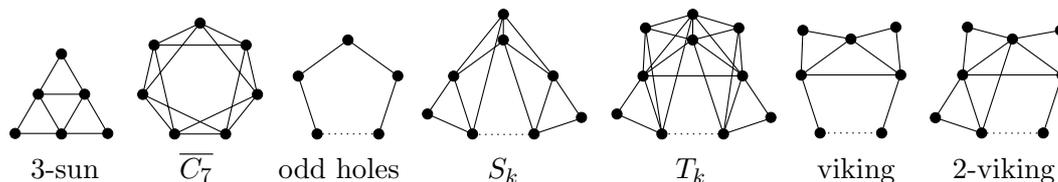


Fig. 2. Minimal forbidden subgraphs for clique-perfect graphs inside the class of HCA graphs. Dotted lines replace any induced path of odd length at least 1.

In fact, we show that an HCA graph that does not contain any of the graphs of Figure 2 is K -perfect. In general, the class of clique-perfect graphs is neither a subclass nor a superclass of the class of K -perfect graphs. But the K -perfection allows us to use arguments similar to those used in the proof of Proposition 12, in order to prove Theorem 9 for HCA graphs that are also HCH . The graphs in $HCA \setminus HCH$ are handled separately.

We start with some easy results about HCH and HCA graphs.

Theorem 13 [8] *Let G be an HCH graph such that $K(G)$ is not perfect.*

- (1) *If $K(G)$ contains $\overline{C_7}$ as induced subgraph, then G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain $\overline{C_7}$, and such that $K(H) = \overline{C_7}$.*
- (2) *If $K(G)$ contains C_{2k+1} as induced subgraph, for some $k \geq 2$, then G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain C_{2k+1} , and such that $K(H) = C_{2k+1}$.*

In this section we will call a *sector* an arc of a circle defined by two points, in order to distinguish them from arcs corresponding to vertices of an HCA graph. For example, in Figure 3, the bold arc is one of the two sectors defined by the points a and b . Given a collection \mathcal{C} of pairwise distinct points on the circle, for $a, b, c \in \mathcal{C}$ we say that c is \mathcal{C} -between a and b if the sector defined by a and b that contains c does not contain any other point of \mathcal{C} . For example, in Figure 3, c is $\{a, b, c, d, e\}$ -between a and b but d is not.

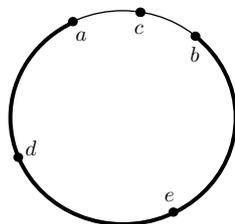


Fig. 3. Example of notation. The bold arc is one of the two *sectors* defined by the points a and b of the circle. c is $\{a, b, c, d, e\}$ -between a and b but d is not.

Lemma 14 *Let G be an HCA graph that has an HCA representation with no two arcs covering the circle. Then G is HCH .*

PROOF. Suppose not. By Theorem 11, G contains a 0-, 1-, 2-, or 3-pyramid P . Let $\{v_1, \dots, v_6\}$ be the vertices of P , such that v_1, v_2, v_3 form a triangle; v_4 is adjacent to v_2 and v_3 but not to v_1 ; v_5 is adjacent to v_1 and v_3 but not to v_2 ; v_6 is adjacent to v_1 and v_2 but not to v_3 . Since P is an induced subgraph of G , P has an HCA representation with no two arcs covering the circle. Let $\mathcal{A} = \{A_i\}_{1 \leq i \leq 6}$ be such a representation, where the arc A_i corresponds to the

vertex v_i . The sets $C_1 = \{v_1, v_2, v_3\}$ and $C_2 = \{v_1, v_2, v_6\}$ are cliques of P , let a be an anchor of C_1 and b of C_2 . Then a and b are distinct points of the circle. Let S_1 and S_2 be the two sectors with ends a, b . Since A_1, A_2 do not cover the circle, and a, b belong to both A_1 and A_2 , we may assume that S_1 is included both in A_1 and in A_2 . Since $a \in A_3$ but $b \notin A_3$, it follows that A_3 has an endpoint, say c , in $S_1 \setminus \{b\}$ (see Figure 4). But now, since the pairs A_1, A_3 and A_2, A_3 do not cover the circle, it follows that either $A_1 \cap A_3 \subseteq A_2$, or $A_2 \cap A_3 \subseteq A_1$. In the former case there is no anchor for the clique $\{v_1, v_3, v_5\}$, and in the later there is none for the clique $\{v_2, v_3, v_4\}$; in both cases a contradiction. \square

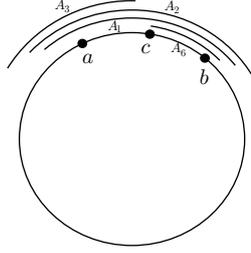


Fig. 4. Scheme of representation of arcs A_6, A_1, A_2 and A_3 , in the proof of Lemma 14.

Lemma 15 *Every HCA representation of a 4-wheel has two arcs covering the circle.*

PROOF. Let a_1, a_2, a_3, a_4, b be the vertices of a 4-wheel W , where $a_1 a_2 a_3 a_4 a_1$ is a cycle of length four and b is adjacent to all of a_1, a_2, a_3, a_4 , and let $\mathcal{A} = \{A_1, A_2, A_3, A_4, B\}$ be an HCA representation of W . Let p_1, p_2, p_3 and p_4 be anchors of the cliques $\{a_1, a_2, b\}$, $\{a_2, a_3, b\}$, $\{a_3, a_4, b\}$, $\{a_4, a_1, b\}$, respectively. Then there are only two possible circular orders of the anchors: p_1, p_2, p_3, p_4 and the reverse one, and for $1 \leq i \leq 4$, each arc A_i passes exactly through p_i and p_{i-1} (index operations are done modulo 4). Since the arc B passes through the four points p_i , it follows that B and one of the A_i cover the circle. \square

Let S denote the unit circle. Let G be an HCA graph that has an HCA representation with no two arcs covering S , and let \mathcal{A} be such a representation. Let H be a clique subgraph of G . Let M_1, \dots, M_s be the cliques of H , and for $1 \leq i \leq s$ let a_i be an anchor of M_i in \mathcal{A} . Let $\varepsilon = \frac{1}{3} \min_{1 \leq i < j \leq s} \text{dist}(a_i, a_j)$, where $\text{dist}(a_i, a_j)$ denotes the length of the shortest sector of S between a_i and a_j . For an arc $A \in \mathcal{A}$ that contains at least one of the points a_1, \dots, a_s , let the *derived arc* A' of A be defined as follows: let a_{i_k}, \dots, a_{i_m} be the points of a_1, \dots, a_s traversed by A in clockwise order, let u be the point of S which is at distance ε from a_{i_k} going anti-clockwise, and v the point of S which is at distance ε from a_{i_m} going clockwise. Let A' be the arc with endpoints u and v and containing all of a_{i_k}, \dots, a_{i_m} . Denote by \mathcal{A}' the set of all arcs A' that are

the derived arcs of some $A \in \mathcal{A}$ such that A contains at least one of a_1, \dots, a_s . Please note that \mathcal{A}' depends on the choice of the anchors a_1, \dots, a_s .

Lemma 16 *Let G be an HCA graph that has an HCA representation with no two arcs covering S , and let \mathcal{A} be such a representation. Let H be a clique subgraph of G . Let M_1, \dots, M_s be the cliques of H , and for $1 \leq i \leq s$ let a_i be an anchor of M_i in \mathcal{A} . Then H is HCA and \mathcal{A}' (as defined above) is an HCA representation of H with no two arcs covering S .*

PROOF. Let H' be the intersection graph of the arcs of \mathcal{A}' . We claim that H' is isomorphic to H . Since the arcs of \mathcal{A}' are sub-arcs of the arcs of \mathcal{A} that correspond to vertices of G that belong to $\bigcup_{i=1}^s M_i$, there is a one-to-one correspondence between the vertices of H' and the vertices of H , and we may assume that $V(H) = V(H')$. Moreover, for every clique M_i and every $A \in \mathcal{A}$, the derived arc of A contains a_i if and only if A does. So M_1, \dots, M_s are cliques on H' , and a_i is an anchor of M_i . Since two vertices of a graph are adjacent if and only if there exists a clique containing them both, in order to show that H is isomorphic to H' , it remains to check that every two adjacent vertices of H' belong to M_i for some i . But it follows from the construction of \mathcal{A}' (and in particular from the choice of ε) that $A'_1 \cap A'_2 \neq \emptyset$ for $A'_1, A'_2 \in \mathcal{A}'$, if and only if $a_i \in A'_1 \cap A'_2$ for some $1 \leq i \leq s$, which means that the corresponding vertices of H' belong to the clique M_i . This proves that $E(H) = E(H')$ and completes the proof of the lemma. \square

An example of the construction of Lemma 16 can be seen in Figure 5.

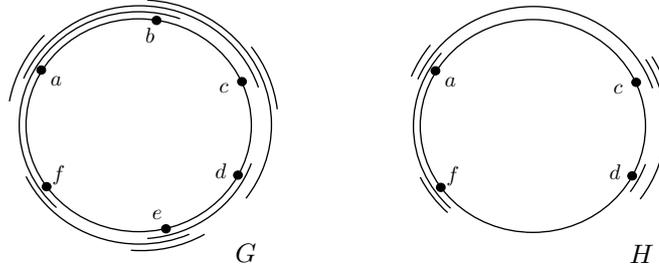


Fig. 5. HCA representation of the clique subgraph H of G whose cliques are a, c, d and f .

Remark 17 *Let G be an HCA graph with representation \mathcal{A} , and let H be a clique subgraph of G with representation \mathcal{A}' given by Lemma 16, with anchors a_1, \dots, a_s . Let $A'_1, A'_2 \in \mathcal{A}'$ be the derived arcs of $A_1, A_2 \in \mathcal{A}$. Then $A_1 \cap A_2$ may be non-empty even if A'_1, A'_2 are disjoint, but no point of $A_1 \setminus A'_1$ or $A_2 \setminus A'_2$ belongs to $\{a_1, \dots, a_s\}$.*

Lemma 18 *Let G be an HCA graph and let \mathcal{A} be an HCA representation of G . Let M_1, \dots, M_k , with $k \geq 5$, be a set of cliques of G such that $M_i \cap M_{i+1}$ is*

non-empty for $i = 1, \dots, k$, and $M_i \cap M_j$ is empty for $j \neq i, i+1, i-1$ (index operations are done modulo k). Let $S = \{v_1, \dots, v_k\}$ such that $v_i \in M_{i-1} \cap M_i$. Let $w \in M_i \setminus S$ non-adjacent to v_{i+2} . Then the neighbors of w in S are either $\{v_i, v_{i+1}\}$, or $\{v_{i-1}, v_i, v_{i+1}\}$, or $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}\}$.

PROOF. For $1 \leq i \leq k$ let m_i be an anchor of M_i , let A_i be the arc of \mathcal{A} corresponding to v_i , and let W be the arc corresponding to w . Since for every i , A_i contains m_{i-1} and m_i , and no m_j with $j \neq i-1, i$, it follows that there are only two possible circular orders of the anchors: m_1, m_2, \dots, m_k and the reverse one. Since w belongs to M_i , it is adjacent to v_i and v_{i+1} , and $m_i \in W$. Since w is non-adjacent to v_{i+2} , w does not belong to M_{i+1} , and $m_{i+1} \notin W$. Since $w \in M_i$ and M_i is disjoint from M_j for $j \neq i-1, i, i+1$, it follows that $m_j \notin W$ for $j \neq i-1, i$ (see Figure 6). Now, if $m_{i-1} \notin W$, then the neighbors of w in S are v_i and v_{i+1} or v_{i-1}, v_i, v_{i+1} , and if $m_{i-1} \in W$, then the neighbors of w in W are v_{i-1}, v_i, v_{i+1} or $v_{i-2}, v_{i-1}, v_i, v_{i+1}$. \square

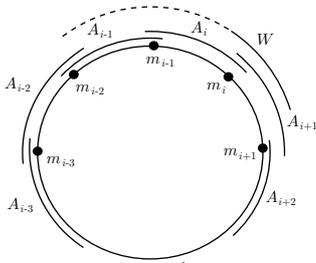


Fig. 6. Scheme of representation of arcs A_{i-3}, \dots, A_{i+2} and W , in the proof of Lemma 18.

In the next theorem we give a sufficient condition for the clique graph of an *HCA* graph to be perfect.

Theorem 19 *Let G be an HCA graph. If G does not contain any of the graphs in Figure 2, then $K(G)$ is perfect.*

PROOF. Let G be an *HCA* graph which does not contain any of the graphs in Figure 2, and \mathcal{A} be an *HCA* representation of G . Assume first that there are two arcs $A_1, A_2 \in \mathcal{A}$ covering the circle, and let v_1, v_2 be the corresponding vertices of G . Then the clique-transversal number of G is at most two, because every anchor of a clique of G is contained in one of A_1, A_2 , and therefore every clique contains either v_1 or v_2 . Since, by Lemma 10, the clique covering number of $K(G)$ is less or equal to the clique-transversal number of G , $K(G)$ is the complement of a bipartite graph, and so it is perfect.

So we may assume no two arcs of \mathcal{A} cover the circle, and, since the representation is *HCA*, no three arcs of \mathcal{A} cover the circle. By Lemma 14, G is

HCH , so $K(G)$ is also HCH [2]. Consequently, if $K(G)$ is not perfect, then it contains an odd hole or $\overline{C_7}$ (for every antihole of length at least eight contains a 2-pyramid, and therefore is not HCH by Theorem 11).

Suppose first that $K(G)$ contains $\overline{C_7}$. By Theorem 13, G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain $\overline{C_7}$. Consider the HCA representation \mathcal{A}' of H given by Lemma 16, and let v_1, \dots, v_7 be vertices inducing $\overline{C_7}$ in H , where the cliques are $\{v_1, v_3, v_5\}$, $\{v_3, v_5, v_7\}$, $\{v_5, v_7, v_2\}$, $\{v_7, v_2, v_4\}$, $\{v_2, v_4, v_6\}$, $\{v_4, v_6, v_1\}$ and $\{v_6, v_1, v_3\}$. That is essentially the unique circular order of the cliques (the other possible order is the reverse one), so the arcs A_1, \dots, A_7 corresponding to v_1, \dots, v_7 must appear in \mathcal{A}' as in Figure 7.

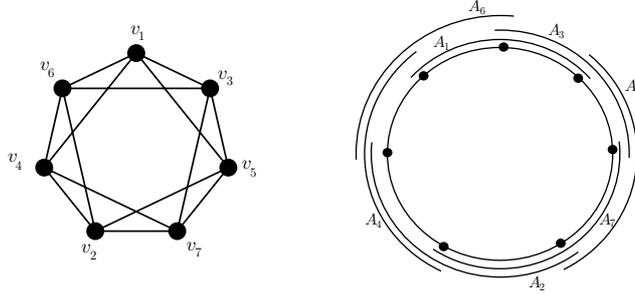


Fig. 7. HCA representation of $\overline{C_7}$.

If some pair of non adjacent vertices v_i, v_j in H are adjacent in G , then there are three arcs covering the circle in \mathcal{A} , a contradiction. Otherwise $\{v_1, \dots, v_7\}$ induce $\overline{C_7}$ in G , a contradiction.

Next suppose that $K(G)$ contains C_{2k+1} , for some $k \geq 2$. By Theorem 13, G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain C_{2k+1} , and such that $K(H) = C_{2k+1}$. Consider the HCA representation \mathcal{A}' of H given by Lemma 16 corresponding to anchors a_1, \dots, a_{2k+1} , and let v_1, \dots, v_{2k+1} be vertices inducing C_{2k+1} in H , where the cliques are $v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $v_n v_1$. Then in G the graph induced by v_1, \dots, v_{2k+1} is a cycle, say C , with chords. Please note that, since H may contain twins that are not twins in G , there may be more than one way to select v_1, \dots, v_{2k+1} . We assume that v_1, \dots, v_{2k+1} are chosen to minimize the number N of chords of C . Again, that is essentially the unique circular order of the cliques (the other possible order is the reverse one), so the arcs A'_1, \dots, A'_{2k+1} corresponding to v_1, \dots, v_{2k+1} must appear in \mathcal{A}' as in Figure 8.

Now it is possible that two disjoint arcs $A'_i, A'_j \in \mathcal{A}'$ are derived from arcs $A_i, A_j \in \mathcal{A}$ whose intersection is non-empty, but it follows from Remark 17 that in this case $|j - i| = 2$ (throughout this proof, indices of vertices in a cycle should be read modulo the length of the cycle). The proof now breaks into cases depending on the values of k and N .

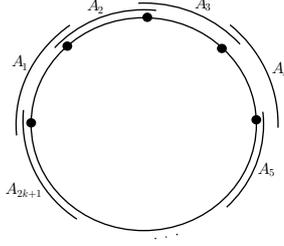


Fig. 8. *HCA* representation of C_{2k+1} , $k \geq 2$.

Case $k = 2$:

Since there are no three arcs in \mathcal{A} covering the circle, C has at most one chord incident with each vertex and so $N \leq 2$. The possible *HCA*-representations of the subgraph of G induced by $\{v_1, \dots, v_5\}$ are depicted in Figure 9. Let M_1, \dots, M_5 be the cliques of H such that M_1 contains v_1 and v_2 , M_2 contains v_2 and v_3 , \dots , M_5 contains v_5 and v_1 , for $1 \leq i \leq 5$, a_i is an anchor of M_i , and the vertices corresponding to M_1, M_2, \dots, M_5 induce C_5 in $K(G)$. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$.

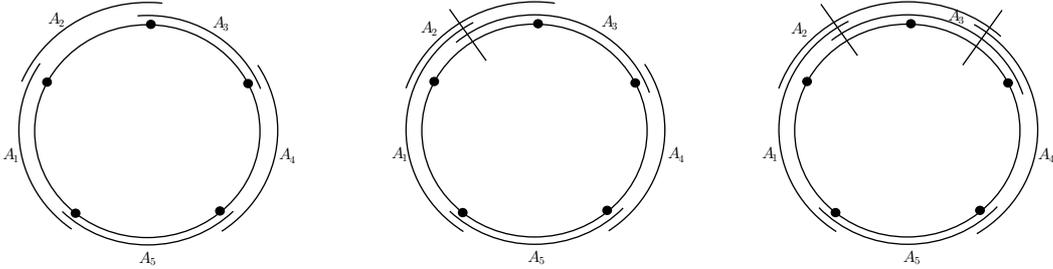


Fig. 9. Possible cases for $k = 2$, corresponding to no chords, one chord or two chords in the cycle.

1. $\underline{N=0}$: In this case G contains an odd hole, a contradiction.
2. $\underline{N=1}$: Suppose that the vertices v_1 and v_3 are adjacent in G . As v_3 does not belong to M_1 , there is a vertex w_1 in M_1 which is not adjacent to v_3 . Analogously, there is a vertex w_2 in M_2 which is not adjacent to v_1 . The vertices w_1 and w_2 are non-adjacent, otherwise v_1, v_3, w_2, w_1, v_2 induce a 4-wheel, which does not have an *HCA* representation with no two arcs covering the circle (Lemma 15). For $i = 1, 2$, w_i can have two, three or four neighbors in C .
 - 2.1. If w_1 and w_2 have two neighbors each one, then $\{v_1, v_2, v_3, v_4, v_5, w_1, w_2\}$ induce a viking.
 - 2.2. If w_1 and w_2 have four neighbors each one, then $\{v_1, w_2, w_1, v_3, v_5, v_2, v_4\}$ induce $\overline{C_7}$.
 - 2.3. If one of w_1, w_2 has three neighbors, say w_1 , for the other case is symmetric, then it follows from Lemma 18 that w_1 is adjacent to v_5, v_1, v_2 . But now $\{w_1, v_2, v_3, v_4, v_5\}$ induce C_5 .

- 2.4. If one of w_1, w_2 has two neighbors and the other one has four neighbors, we may assume that w_1 has two and w_2 has four (the other case is symmetric). The clique M_4 does not intersect M_2 , so w_2 does not belong to M_4 and there is a vertex w_3 in M_4 which is not adjacent to w_2 .

If the arcs corresponding to w_3 and v_3 intersect in a point of the circle that is A -between a_3 and a_4 , then one of them passes through a point that belongs both to the arc corresponding to v_5 and to the arc corresponding to w_2 , but w_3 is non-adjacent to w_2 and v_3 is non-adjacent to v_5 , a contradiction. If the arcs corresponding to w_3 and v_3 intersect in a point of the circle A -between a_1 and a_2 , then the arcs corresponding to v_3, v_4 and w_3 cover the circle. So w_3 and v_3 are not adjacent, and w_3 can be adjacent either to v_4, v_5, v_1 and v_2 ; or to v_4, v_5 and v_1 ; or only to v_4 and v_5 . In the first case, the vertices $v_1, w_2, w_3, v_3, v_5, v_2, v_4$ induce $\overline{C_7}$. In the second case, the vertices v_1, v_2, w_2, v_4, w_3 induce C_5 . In the last case, the eight vertices induce S_2 .

3. $\underline{N=2}$: The same vertex cannot belong to two chords, so all the cases are symmetric to the case where v_1 is adjacent to v_3 and v_2 to v_4 . As v_3 does not belong to M_1 , there is a vertex w_1 in M_1 which is not adjacent to v_3 . Analogously, as v_2 does not belong to M_3 , there is a vertex w_3 in M_3 which is not adjacent to v_2 .

Please note that if w_3 is adjacent to v_1 then their corresponding arcs must intersect in a point of the circle A -between a_4 and a_5 , because w_3 is not adjacent to v_2 . But in this case the arcs corresponding to v_1, v_3 and w_3 cover the circle, so w_3 is not adjacent to v_1 . Analogously, we can prove that w_1 is not adjacent to v_4 .

- 3.1. If w_1 and w_3 are adjacent, then their corresponding arcs must intersect in a point of the circle A -between a_4 and a_5 , because w_1 is non-adjacent to v_3 and v_4 and w_3 is non-adjacent to v_1 and v_2 . So both are adjacent to v_5 , and the vertices $v_1, v_4, w_1, v_3, v_5, v_2, w_3$ induce $\overline{C_7}$.
- 3.2. If w_1 and w_3 are not adjacent but both of them are adjacent to v_5 , the vertices w_1, v_2, v_3, w_3, v_5 induce C_5 .
- 3.3. The remaining case is when w_1 and w_3 are not adjacent but at most one of them is adjacent to v_5 .

For this case, we have to consider the clique M_2 . Since v_1 and v_4 do not belong to M_2 , there is a vertex in M_2 which is not adjacent to v_1 and there is a vertex in M_2 which is not adjacent to v_4 .

- 3.3.1. If there is a vertex w which is non-adjacent to v_1 and v_4 , then w cannot be adjacent either to w_1 or w_3 , otherwise v_1, v_3, w, w_1, v_2 (or v_2, w, w_3, v_4, v_3 , respectively) induce a 4-wheel, a contradiction by Lemma 15.

Therefore, if each of w_1 and w_3 has two neighbors in C , then the vertices $v_1, \dots, v_5, w_1, w, w_3$ induce a 2-viking in G , and,

if w_1 and w_3 have two and three neighbors (respectively) in C , the vertices $v_1, v_2, v_3, w_3, v_5, w_1, w$ induce a viking in G (the case when w_1 has three neighbors and w_3 has two neighbors in C is symmetric).

3.3.2. If there is no such a vertex w , every vertex of M_2 is either adjacent to v_1 or to v_4 . Then there exist two vertices w_2 and w_4 in M_2 , such that w_2 is adjacent to v_4 but not to v_1 and w_4 is adjacent to v_1 but not to v_4 . Since by Lemma 15 G does not contain a 4-wheel, it follows that w_2 is not adjacent to w_1 and w_4 is not adjacent to w_3 . If neither w_4 nor w_2 is adjacent to v_5 , then the vertices v_1, w_4, w_2, v_4, v_5 induce C_5 . If w_2 and w_4 are both adjacent to v_5 , then the arcs corresponding to w_2, w_4 and v_5 cover the circle. Otherwise, suppose w_2 is adjacent to v_5 and w_4 is not (the other case is symmetric), so by the circular-arc representation w_2 belongs to M_3 , and it is adjacent to w_3 .

In this case w_2 is a twin of v_3 in H . Consider the hole formed by $\{v_1, v_2, w_2, v_4, v_5\}$ in H , say C' . In G , $\{v_1, v_2, w_2, v_4, v_5\}$ induces a cycle with two chords, v_2v_4 and w_2v_5 . If vertex w_3 has only two neighbors in C , then it has two neighbors in C' , namely w_2 and v_4 , and it is non-adjacent to v_2 and v_5 , so we get a contradiction by a previous case (Case 3.3.1).

The last case is when w_3 has three neighbors in C and w_1 has only two. If w_3 belongs to M_4 then w_3 and v_4 are twins in H , but the cycle of H obtained by replacing v_4 with w_3 in C has only one chord in G , contrary to the choice of C .

If w_3 does not belong to M_4 , let w_5 be a vertex of M_4 , that minimizes the distance of the endpoint of its corresponding arc that lies A -between a_3 and a_4 , to a_4 . Since none of w_2, v_3, w_3 belongs to M_4 , they are not adjacent to w_5 . The set of neighbors of w_5 in C includes $\{v_4, v_5\}$ and, by Lemma 18, is a subset of $\{v_1, v_2, v_4, v_5\}$. If w_5 is adjacent to v_1 and v_2 , then the arcs corresponding to vertices v_2, v_4 and w_5 cover the circle. If w_5 is adjacent to v_1 but not to v_2 , then the vertices v_1, w_4, w_2, v_4, w_5 induce C_5 . If w_5 has only two neighbors in C (v_4 and v_5), then w_1 and w_5 are non-adjacent, because w_1 is non-adjacent to v_5 and w_5 is non-adjacent to v_1 . Now if w_4 and w_1 are non-adjacent, then the vertices $\{v_1, \dots, v_5, w_1, \dots, w_5\}$ induce T_2 , otherwise, the eight vertices $v_1, w_4, v_3, v_4, v_5, w_1, w_2, w_5$ induce S_2 .

Case $k \geq 3$: Let M_1, \dots, M_{2k+1} be the cliques of H such that M_1 contains v_1 and v_2 , M_2 contains v_2 and v_3 , \dots , M_{2k+1} contains v_{2k+1} and v_1 , for $1 \leq i \leq 2k+1$, a_i is an anchor of M_i , and the vertices corresponding to $M_1, M_2, \dots, M_{2k+1}$ induce C_{2k+1} in $K(G)$. Let $A = \{a_1, \dots, a_{2k+1}\}$. We remind the reader that if v_i is adjacent to v_j in G , then $|i - j| \leq 2$.

If $N = 0$, then G contains an odd hole, one of the forbidden subgraphs of Figure 2. If $N = 1$, say v_1v_3 is a chord of C , then the arcs corresponding to v_1 and v_3 intersect in some point of the circle that is A -between a_1 and a_2 . The vertices v_1, v_2 and v_3 belong to some clique M of G , distinct from M_i for $i = 1, \dots, 2k + 1$. Every anchor of M is A -between a_1 and a_2 , every vertex of M which is not in H is only adjacent to vertices of H belonging to M_1 or M_2 (their corresponding arcs are bounded by a_1 and a_2), and every vertex of M in H belongs to M_1 or M_2 . Both M_1 and M_2 are disjoint from M_4, \dots, M_{2k} , so M is disjoint from M_4, \dots, M_{2k} . But the vertex v_1 belongs to $M \cap M_{2k+1}$ and vertex v_3 belongs to $M \cap M_3$, and therefore $M, M_3, M_4, \dots, M_{2k}, M_{2k+1}$ induce C_{2k} in $K(G)$.

Repeating this argument twice (we do not use the fact that the cycle is odd, but only the fact that it has at least six vertices), if there exist two chords $v_i v_{i+2}$ and $v_j v_{j+2}$ in C such that $v_i v_{i+1}, v_{i+1} v_{i+2}, v_j v_{j+1}$ and $v_{j+1} v_{j+2}$ are four distinct edges of G , we can reduce the problem to a smaller one, the case of an odd hole with $2k - 1$ vertices induced in $K(G)$.

So we only need to consider two cases:

- $N = 1$; and
 - $N = 2$, and for some i , v_i is adjacent to v_{i+2} and v_{i+1} is adjacent to v_{i+3} .
1. $N=1$: Suppose that the vertices v_1 and v_3 are adjacent in G . As v_3 does not belong to M_1 , there is a vertex w_1 in M_1 which is not adjacent to v_3 . Analogously, there is a vertex w_2 in M_2 which is not adjacent to v_1 . The vertices w_1 and w_2 are non-adjacent, otherwise $\{v_1, v_3, w_2, w_1, v_2\}$ induces a 4-wheel, contrary to Lemma 15. By Lemma 18 the vertex w_1 has two, three or four neighbors in C and they are consecutive in it (v_2 and v_1 ; or v_2, v_1 and v_{2k+1} ; or v_2, v_1, v_{2k+1} and v_{2k} , respectively). Analogously, w_2 has two, three or four neighbors in C and they are consecutive in the cycle (v_2 and v_3 ; or v_2, v_3 and v_4 ; or v_2, v_3, v_4 and v_5 , respectively). In all cases w_1 and w_2 have no common neighbors in $V(C) \setminus \{v_2\}$, since $k \geq 3$.
 - 1.1. If w_1 and w_2 have exactly two neighbors each one in C , the vertices $v_1, \dots, v_{2k+1}, w_1, w_2$ induce a viking.
 - 1.2. If w_1 and w_2 have exactly four neighbors each one in C , the vertices $w_1, v_2, w_2, v_5, \dots, v_{2k}$ induce C_{2k-1} .
 - 1.3. If one of w_1, w_2 has exactly three neighbors in C (suppose w_1 , the other case is symmetric), the vertices $w_1, v_2, v_3, \dots, v_{2k+1}$ induce C_{2k+1} .
 - 1.4. If one of w_1, w_2 has exactly two neighbors in C and the other one has exactly four neighbors in C , suppose w_1 has two and w_2 has four (the other case is symmetric). The clique M_4 is disjoint from M_2 , so w_2 does not belong to M_4 and there is a vertex w_3 in M_4 which is not adjacent to w_2 .

The arc corresponding to w_3 cannot pass through the points of the circle corresponding either to M_3 (because w_2 and w_3 are not adjacent) or to M_6 (because M_4 and M_6 are disjoint), so if the arcs corresponding to w_3 and v_3 have non-empty intersection, they must intersect at a point of the circle that is A -between a_3 and a_4 . In this case one of them passes through a point that belongs to both the arc corresponding to v_5 and the arc corresponding to w_2 , but w_3 is non-adjacent to w_2 , and v_3 is non-adjacent to v_5 . So w_3 and v_3 are not adjacent, and, by Lemma 18, w_3 can be adjacent either to v_4, v_5, v_6 and v_7 ; or to v_4, v_5 and v_6 ; or only to v_4 and v_5 . In the first case, the vertices $v_1, v_3, v_4, w_3, v_7, \dots, v_{2k+1}$ induce C_{2k-1} . In the second case, the vertices $v_1, v_2, w_2, v_4, w_3, v_6, \dots, v_{2k+1}$ induce C_{2k+1} . In the last case, the $2k+4$ vertices $v_1, \dots, v_{2k+1}, w_1, w_2, w_3$ induce S_k .

2. $N=2$, and for some i , v_i is adjacent to v_{i+2} and v_{i+1} is adjacent to v_{i+3} :
Without loss of generality, we may assume that $i = 1$, so the chords are v_1v_3 and v_2v_4 . As v_3 does not belong to M_1 , there is a vertex w_1 in M_1 which is not adjacent to v_3 . As v_2 does not belong to M_3 , there is a vertex w_3 in M_3 which is not adjacent to v_2 . No vertex of G belongs to more than two cliques of M_1, \dots, M_{2k+1} . These facts imply that the vertices w_1 and w_3 are non-adjacent, and, by Lemma 18, each of them has two, three or four consecutive neighbors in C . The vertex w_3 can be adjacent to v_3, v_4, v_5 and v_6 ; or to v_3, v_4 and v_5 ; or only to v_3 and v_4 . The vertex w_1 can be adjacent to v_2, v_1, v_{2k+1} and v_{2k} ; or to v_2, v_1 and v_{2k+1} ; or only to v_2 and v_1 .
- 2.1. If w_3 has four neighbors in C , then the vertices $v_1, v_3, w_3, v_6, \dots, v_{2k+1}$ induce C_{2k-1} . The case of w_1 having four neighbors is symmetric.
- 2.2. If w_1 and w_3 have three neighbors each one in C , then the vertices $w_1, v_2, v_3, w_3, v_5, \dots, v_{2k+1}$ induce C_{2k+1} .
- 2.3. It remains to analyze the cases when w_1 and w_3 each have two neighbors in C , and when one of them has three neighbors in C and the other one has two. For these cases, we have to consider the clique M_2 .
Since v_1 and v_4 do not belong to M_2 , there is a vertex in M_2 which is not adjacent to v_1 and there is a vertex in M_2 which is not adjacent to v_4 .
- 2.3.1. If there is a vertex $w \in M_2$ which is non-adjacent to v_1 and v_4 , then w is non-adjacent to w_1 and w_3 , for otherwise $\{v_1, v_3, w, w_1, v_2\}$ (or $\{v_2, w, w_3, v_4, v_3\}$, respectively) induces a 4-wheel, contrary to Lemma 15.

Therefore, if w_1 and w_3 have two neighbors each in C , then the vertices $v_1, \dots, v_{2k+1}, w_1, w, w_3$ induce a 2-viking in G . If w_1 and w_3 have two and three neighbors (respectively) in C , then $v_1, v_2, v_3, w_3, v_5, \dots, v_{2k+1}, w_1, w$ induce a viking in G . If w_1 has three neighbors and w_3 has two neighbors in C , then $w_1, v_2, v_3, \dots, v_{2k+1}, w, w_3$ induce a viking in G .

2.3.2. If no such a vertex w exists, then every vertex of M_2 is either adjacent to v_1 or to v_4 , and there exist two vertices w_2 and w_4 in M_2 , such that w_2 is adjacent to v_4 but not to v_1 and w_4 is adjacent to v_1 but not to v_4 . Since G does not contain a 4-wheel, it follows that w_2 is not adjacent to w_1 and w_4 is not adjacent to w_3 . If w_4 is not adjacent to v_{2k+1} and w_2 is not adjacent to v_5 , then the vertices $v_1, w_4, w_2, v_4, \dots, v_{2k+1}$ induce C_{2k+1} . If w_4 is adjacent to v_{2k+1} and w_2 is adjacent to v_5 , then the vertices $w_4, w_2, v_5, \dots, v_{2k+1}$ induce C_{2k-1} . Otherwise, suppose w_2 is adjacent to v_5 and w_4 is not adjacent to v_{2k+1} (the other case is symmetric), so since G is a circular-arc graph, w_2 belongs to M_3 , and it is adjacent to w_3 . In this case w_2 is a twin of v_3 in H . Consider the hole $\{v_1, v_2, w_2, v_4, \dots, v_{2k+1}\}$, say C' , in H . The graph induced by $\{v_1, v_2, w_2, v_4, \dots, v_{2k+1}\}$ in G is a cycle with two chords, v_2v_4 and w_2v_5 . If the vertex w_3 has exactly two neighbors in C , then it has exactly two neighbors in C' , namely w_2 and v_4 , and it is non-adjacent to v_2 and v_5 , and we get a contradiction by a previous case (Case 2.3.1).

The last case is when w_3 has three neighbors in the cycle and w_1 has only two. If w_3 belongs to M_4 then w_3 and v_4 are twins in H , but the cycle of H obtained by replacing v_4 with w_3 in C has only one chord in G , contrary to the choice of C .

If w_3 does not belong to M_4 , let w_5 be a vertex of M_4 , that minimizes the distance of the endpoint of its corresponding arc that lies A -between a_3 and a_4 , to a_4 . Since w_2, v_3, w_3 do not belong to M_4 , they are not adjacent to w_5 . The neighbor set of the vertex w_5 includes $\{v_4, v_5\}$ and, by Lemma 18, is a subset of $\{v_4, v_5, v_6, v_7\}$. If w_5 is adjacent to v_6 and v_7 , then the vertices $v_1, v_3, v_4, w_5, v_7, \dots, v_{2k+1}$ induce C_{2k-1} . If w_5 is adjacent to v_6 but not to v_7 , then the vertices $v_1, w_4, w_2, v_4, w_5, v_6, \dots, v_{2k+1}$ induce C_{2k+1} . So we may assume that v_4 and v_5 are the only neighbors of w_5 in C . But now, if w_4 and w_1 are not adjacent, then the vertices $v_1, \dots, v_{2k+1}, w_1, \dots, w_5$ induce T_k , and otherwise, the $2k + 4$ vertices $v_1, w_4, v_3, \dots, v_{2k+1}, w_1, w_2, w_5$ induce S_k .

In each case we get a contradiction. This concludes the proof. \square

We can now prove the characterization theorem for HCA graphs.

Proof of Theorem 9. The “only if” part follows from Theorem 1, Proposition 2 and Proposition 3. Let us prove the “if” statement. Let G be an HCA graph which does not contain any of the graphs in Figure 2, and let \mathcal{A} be an HCA representation of G . Since the class of HCA graph is hereditary, it is

enough to prove that $\tau_c(G) = \alpha_c(G)$.

Assume first that some two arcs of \mathcal{A} cover the circle. Then $\tau_c(G) \leq 2$. If $\tau_c(G) = 1$ or $\alpha_c(G) = 2$, then $\alpha_c(G) = \tau_c(G)$ and the theorem holds. So we may assume that $\tau_c(G) = 2$ and $\alpha_c(G) = 1$. By Theorem 4, G contains Q_{2k+1} for some $k \geq 1$. It is not difficult to check that the 3-pyramid is not an *HCA* graph. Moreover, $\overline{C_{2k+1}}$ (an induced subgraph of Q_{2k+1}) contains the 3-pyramid for $k \geq 4$. So, G contains either Q_3 , or Q_5 , or Q_7 . But Q_3 is the 3-sun, Q_5 contains C_5 and Q_7 contains $\overline{C_7}$, a contradiction.

So we may assume that no two arcs of \mathcal{A} cover the circle. But now, by Lemma 14 and Theorem 19, G is clique-Helly and K-perfect, and so, by Lemma 10, $\tau_c(G) = \alpha_c(G)$. \square

It is easy to check that no two graphs of the families represented in Figure 2 are properly contained in each other. Therefore, as a corollary of Theorem 9, we obtain the following result.

Corollary 20 *Vikings, 2-vikings, S_k and T_k ($k \geq 2$), are minimally clique-imperfect.*

4.1 Recognition algorithm

Helly circular-arc graphs can be recognized in polynomial time [19] and, given a Helly representation of an *HCA* graph G , both parameters $\tau_c(G)$ and $\alpha_c(G)$ can be computed in linear time [16,17]. However, these properties do not immediately imply the existence of a polynomial time recognition algorithm for clique-perfect *HCA* graphs (we need the equality for every induced subgraph). The characterization in Theorem 9 leads to such an algorithm, which is strongly based on the recognition of perfect graphs [13]. The idea of the algorithm is similar to the one used in [15] for recognizing balanceable matrices.

Algorithm:

Input: An *HCA* graph $G = (V, E)$.

Output: TRUE if G is clique-perfect, FALSE if G is not.

- (1) Check if G contains a 3-sun. If G contains a 3-sun, return FALSE.
- (2) (*Checking for odd holes and $\overline{C_7}$*) Check if G is perfect. If G is not perfect, return FALSE.
- (3) (*Checking for vikings*) For every 7-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2$ such that the edges between those vertices in G are $a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_4a_5, b_1a_2, b_1a_3, b_2a_3, b_2a_4$, and possibly a_1a_5 , do the following:
 - (a) If a_1 is adjacent to a_5 , return FALSE.

- (b) Let G' be the graph obtained from G by removing the vertices a_2, a_3, a_4, b_1, b_2 and all their neighbors except for a_1 and a_5 , and adding a new vertex c adjacent only to a_1 and a_5 .
- (c) Check if G' is perfect. If G' is not perfect, return FALSE.
- (4) (*Checking for 2-vikings*) For every 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$ such that the edges between those vertices in G are $a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_3a_5, a_4a_5, b_1a_2, b_1a_3, b_2a_3, b_2a_4, b_3a_4$ and b_3a_5 , do the following:
 - (a) If a_1 is adjacent to a_5 , return FALSE.
 - (b) Let G' be the graph obtained from G by removing the vertices $a_2, a_3, a_4, b_1, b_2, b_3$ and all their neighbors except for a_1 and a_5 , and adding a new vertex c adjacent only to a_1 and a_5 .
 - (c) Check if G' is perfect. If G' is not perfect, return FALSE.
- (5) (*Checking for S_k*) For every 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$ such that the edges between those vertices in G are $a_1a_2, a_2a_3, a_3a_4, a_3a_5, a_4a_5, b_1a_1, b_1a_2, b_2a_4, b_2a_5, b_3a_1, b_3a_2, b_3a_3, b_3a_4$, and possibly a_1a_5 , do the following:
 - (a) If a_1 is adjacent to a_5 , return FALSE.
 - (b) Let G' be the graph obtained from G by removing the vertices $a_2, a_3, a_4, b_1, b_2, b_3$ and all their neighbors except for a_1 and a_5 , and adding a new vertex c adjacent only to a_1 and a_5 .
 - (c) Check if G' is perfect. If G' is not perfect, return FALSE.
- (6) (*Checking for T_k*) For every 10-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5$ such that the edges between those vertices in G are $a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_3a_5, a_4a_5, b_1a_1, b_1a_2, b_2a_1, b_2a_2, b_2a_3, b_2b_3, b_3a_1, b_3a_2, b_3a_3, b_3a_4, b_3b_4, b_4a_3, b_4a_4, b_4a_5, b_5a_4, b_5a_5$, and possibly a_1a_5 , do the following:
 - (a) If a_1 is adjacent to a_5 , return FALSE.
 - (b) Let G' be the graph obtained from G by removing the vertices $a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5$ and all their neighbors except for a_1 and a_5 , and adding a new vertex c adjacent only to a_1 and a_5 .
 - (c) Check if G' is perfect. If G' is not perfect, return FALSE.
- (7) Return TRUE.

Correctness: The output of the algorithm is TRUE if it finishes in step (7), otherwise the output is FALSE. Let us prove that, given as input an HCA graph G , the algorithm finishes in step (7) if and only if G does not contain the graphs of Figure 2. The correctness of the algorithm then follows from Theorem 9.

Let G be an HCA graph. Step (1) will output FALSE if and only if G contains a 3-sun. So henceforth suppose that G does not contain a 3-sun.

1. *Step (2) will output FALSE if and only if G contains an odd hole or $\overline{C_7}$.*

If G contains an odd hole or $\overline{C_7}$ then it is not perfect. Conversely, if G is not perfect it contains an odd hole or an odd antihole. Since G is HCA , it does not contain an antihole of length at least nine. So G must contain an odd

hole or $\overline{C_7}$. This proves 1. So henceforth suppose that G is perfect, and, in particular, it does not contain an odd hole or $\overline{C_7}$.

2. *Step (3) will output FALSE if and only if G contains a viking.*

If G contains a viking H with $V(H) = \{a_1, \dots, a_{2k+1}, b_1, b_2\}$ and adjacencies as defined in Section 2, at some point the algorithm will consider the 7-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2$. In H , either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (3.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5 a_6 \dots a_{2k+1} a_1$. Since a_6, \dots, a_{2k+1} are non-neighbors of a_2, a_3, a_4, b_1, b_2 , it follows that $ca_5 a_6 \dots a_{2k+1} a_1 c$ is an odd hole in G' , so the algorithm will output FALSE at step (3.c).

Conversely, if the algorithm outputs FALSE at step (3.a), then $\{a_1, \dots, a_5, b_1, b_2\}$ induce a viking in G . If the algorithm outputs FALSE at step (3.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G' . Since it has degree two, c belongs to an odd hole $ca_5 v_1 \dots v_{2t} a_1 c$ in G' . Since v_1, \dots, v_{2t} are different from and non-adjacent to a_2, a_3, a_4, b_1, b_2 , it follows that $\{a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, b_2\}$ induce a viking in G . This proves 2. So henceforth suppose that G contains no viking.

3. *Step (4) will output FALSE if and only if G contains a 2-viking.*

If G contains a 2-viking H with $V(H) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3\}$ and adjacencies as defined in Section 2, at some point the algorithm will consider the 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$. In H , either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (4.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5 a_6 \dots a_{2k+1} a_1$. Since a_6, \dots, a_{2k+1} are non-neighbors of $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $ca_5 a_6 \dots a_{2k+1} a_1 c$ is an odd hole in G' , so the algorithm will output FALSE at step (4.c).

Conversely, if the algorithm outputs FALSE at step (4.a), then $\{a_1, \dots, a_5, b_1, b_2, b_3\}$ induce a 2-viking in G . If the algorithm outputs FALSE at step (4.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G' . Since it has degree two, c belongs to an odd hole $ca_5 v_1 \dots v_{2t} a_1 c$ in G' . Since v_1, \dots, v_{2t} are different from and non-adjacent to $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, b_2, b_3$ induce a 2-viking in G . This proves 3. So henceforth suppose that G contains no 2-viking.

4. *Step (5) will output FALSE if and only if G contains S_k for some $k \geq 2$.*

If G contains S_k for some $k \geq 2$, with $V(S_k) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3\}$ and adjacencies as defined in Section 2, at some point the algorithm will consider the 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$. In S_k , either $k = 2$ and a_1 is adjacent to a_5 (in

this case the algorithm will output FALSE at step (5.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \dots a_{2k+1}a_1$. Since a_6, \dots, a_{2k+1} are non-neighbors of $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $ca_5a_6 \dots a_{2k+1}a_1c$ is an odd hole in G' , so the algorithm will output FALSE at step (5.c).

Conversely, if the algorithm outputs FALSE at step (5.a), then vertices $\{a_1, \dots, a_5, b_1, b_2, b_3\}$ induce S_2 in G . If the algorithm outputs FALSE at step (5.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G' . Since it has degree two, c belongs to an odd hole $ca_5v_1 \dots v_{2t}a_1c$ in G' . Since v_1, \dots, v_{2t} are different from and non-adjacent to $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that vertices $\{a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, b_2, b_3\}$ induce S_{t+2} in G . This proves 4. So henceforth suppose that G does not contain S_k for $k \geq 2$.

5. Step (6) will output FALSE if and only if G contains T_k for some $k \geq 2$.

If G contains T_k for some $k \geq 2$, with $V(T_k) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3, b_4, b_5\}$ and adjacencies as defined in Section 2, at some point the algorithm will consider the 10-tuple $a_1, \dots, a_5, b_1, \dots, b_5$. In T_k , either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (6.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \dots a_{2k+1}a_1$. Since a_6, \dots, a_{2k+1} are non-neighbors of $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $ca_5a_6 \dots a_{2k+1}a_1c$ is an odd hole in G' , so the algorithm will output FALSE at step (6.c).

Conversely, if the algorithm outputs FALSE at step (6.a), then vertices $\{a_1, \dots, a_5, b_1, \dots, b_5\}$ induce S_2 in G . If the algorithm outputs FALSE at step (6.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G' . Since it has degree two, c belongs to an odd hole $ca_5v_1 \dots v_{2t}a_1c$ in G' . Since v_1, \dots, v_{2t} are different from and non-adjacent to $a_2, a_3, a_4, b_1, \dots, b_5$, it follows that $\{a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, \dots, b_5\}$ induce T_{t+2} in G . This proves 5, and completes the proof of correctness. \square

Time complexity: The time complexity of the best known algorithm to recognize perfect graphs is $O(|V|^9)$ [13]. So the time complexity of this algorithm is given by step (6) and it is $O(|V|^{19})$.

Thus we can answer affirmatively the question of the existence of a polynomial time recognition algorithm for clique-perfect graphs within the class of HCA graphs.

5 Summary

These results allow us to formulate partial characterizations of clique-perfect graphs by forbidden subgraphs, as is shown in Table 1.

Graph classes	Forbidden subgraphs	Reference
Diamond-free graphs	odd generalized suns	Thm 8
<i>HCA</i> graphs	graphs in Figure 2	Thm 9

Table 1

Forbidden induced subgraphs for clique-perfect graphs in each studied class.

Note that in the second case all the forbidden induced subgraphs are minimal. In the first case, however, we need to forbid every odd-generalized sun. Obviously, in this case it is enough to forbid diamond-free odd generalized suns. It is easy to see that all such suns have no improper edges but we do not yet know what the minimal diamond-free odd generalized suns are.

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